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ORBIT CARDINALS: ON THE EFFECTIVE CARDINALITIES ARISING AS QUOTIENT SPACES OF THE FORM X/G WHERE G ACTS ON A POLISH SPACE X

BY

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ABSTRACT

We prove an Ulm-type classification theorem for actions in $L(\mathbb{R})$, thereby answering a question of Becker and Kechris, and investigate the effective cardinalities which can be induced by various classes of Polish groups.

0. Introduction

This paper lies at the abstract end of a project to find a structure theory for some very general objects. It considers the **effective cardinalities** that arise from the continuous actions of Polish groups. The philosophy is to calculate cardinalities using only sets and functions that are in some sense **reasonable** or **definable**. As in [19] and [5], the notion of effective cardinalities is intended to measure the relative difficulty of classification problems, in that we may say that the classification of the equivalence relation E on X is harder than the classification of F on Y if the effective cardinality of X/E exceeds that of Y/F.

Of course the notion of reasonable is vague and subject to personal taste and prejudice. I will choose the most generous explication in wide currency. For me, the reasonably definable sets are those that appear in $L(\mathbb{R})$, the universe of all objects that arise from transfinite operations applied to $\mathbb{R}$.

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This may very well be too liberal for some, and an alternative approach would be to restrict ourselves to say the Borel sets, thereby giving us the notion of **Borel cardinality**; alternatively we may diet on the sets and functions arising in the σ-algebra generated by the open sets and closed under continuous images. For most of the problems considered below there is little difference between Borel and $L(\mathbb{R})$. Indeed, under the assumption of $\mathrm{AD}^{L(\mathbb{R})}$, the universe of $L(\mathbb{R})$ continues the sketch presented by the Borel sets, providing a canonical model of ZF where every set of reals has regularity properties such as being Lebesgue measurable and the cardinal structure plays out the suggestions made by the Borel equivalence relations.

It should be stressed that $L(\mathbb{R})$ is a model of ZF, but not of choice. Thus not every set can be wellordered, and consequently not every cardinal corresponds to an ordinal. For instance, the cardinality of $2^{\aleph_0}$ is not an ordinal in $L(\mathbb{R})$— just as there is no Borel wellordering of $\mathbb{R}$ in ZFC. Moreover, the existence of a surjection π: $A \twoheadrightarrow B$ does not guarantee that $|B|$, the cardinality of B, is less than the cardinality of A, in the sense of there being an injection from B to A. For instance, although there is a surjection from $\mathbb{R}$ to $\mathbb{Q}/\mathbb{R}$ in $L(\mathbb{R})$, there is no injection in $L(\mathbb{R})$ from $\mathbb{Q}/\mathbb{R}$ to $\mathbb{R}$. To keep the distinctions in view, I will always write $|A|_{L(\mathbb{R})}$ to indicate the cardinality of A as calculated in $L(\mathbb{R})$.

The first result is one in a long line of generalizations of the Glimm–Effros dichotomy for Polish group actions.

0.1 THEOREM ($\mathrm{AD}^{L(\mathbb{R})}$): *Let G be a Polish group acting continuously on a Polish space X, and let $A \subset X$ be in $L(\mathbb{R})$. Then either*

(I) $|A/G|_{L(\mathbb{R})} \leq |2^{<\omega_1}|_{L(\mathbb{R})}$,

or

(II) $|\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})} \leq |A/G|_{L(\mathbb{R})}$.

Here A/G is the set of all orbits intersecting $A - \{G \cdot a : a \in A\}$.

The proof of 0.1 also works in the $\mathrm{AD}_{\mathbb{R}}$ context, thereby answering a question from [3].

Another direction was suggested by recent work of Howard Becker's:

0.2 THEOREM (Becker): *Let G be a Polish group with a left invariant complete metric acting continuously on a Polish space X. Then either*

(I) *there is a Borel θ: $X \to 2^\omega$ such that for all $x_1, x_2 \in X$*

$$\exists g \in G(g \cdot x_1 = x_2) \Leftrightarrow \theta(x_1) = \theta(x_2),$$

or

(II) *there is a Borel* $\theta\colon \mathbb{R} \to X$ *such that for all* $r_1, r_2 \in \mathbb{R}$

$$r_1 - r_2 \in \mathbb{Q} \Leftrightarrow \exists g \in G(g \cdot \theta(r_1) = \theta(r_2)).$$

The class of Polish groups with a left invariant complete metric includes all locally compact and all solvable Polish groups, but not the symmetric group of permutations on a countably infinite set with the topology of pointwise convergence nor the automorphism group of $[0,1]$ under the compact-open topology.

While Becker also established a weakening of this result for $\underset{\sim}{\Sigma}^1_1$ sets (that is, those arising as the continuous images of Borel sets), he did so only under the additional assumption that **every real has a sharp**. Below we obtain just in ZFC that

0.3 THEOREM: *Let G be a Polish group with a left invariant complete metric acting continuously on a Polish space X, and let $A \subset X$ be $\underset{\sim}{\Sigma}^1_1$. Then either*

(I) *there is a* $\underset{\sim}{\Delta}^1_2$ *function* $\theta\colon A \to 2^\omega$ *such that for all* $x_1, x_2 \in A$

$$\exists g \in G(g \cdot x_1 = x_2) \Leftrightarrow \theta(x_1) = \theta(x_2),$$

or

(II) *there is Borel* $\theta\colon \mathbb{R} \to A$ *such that for all* $r_1, r_2 \in \mathbb{R}$

$$r_1 - r_2 \in \mathbb{Q} \Leftrightarrow \exists g \in G(g \cdot \theta(r_1) = \theta(r_2)).$$

The method of 0.3 gives in ZFC alone that any Polish group G with a left invariant complete metric satisfies Vaught's conjecture on $\underset{\sim}{\Sigma}^1_1$ sets—in that if X is a Polish G-space and $A \subset X$ is analytic, then either A has a perfect subset of E^X_G-inequivalent reals or else $|A/G| \leq \aleph_0$. The proof also yields under appropriate determinacy or large cardinal assumptions a generalization that Becker's arguments do not seem to give under any hypothesis.

0.4 THEOREM ($\mathrm{AD}^{L(\mathbb{R})}$): *Let G be a Polish group with a left invariant complete metric acting continuously on a Polish space X, and let $A \subset X$ be in $L(\mathbb{R})$. Then either*

(I) $|A/G|_{L(\mathbb{R})} \leq |2^\omega|_{L(\mathbb{R})}$,

or

(II) $|\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})} \leq |A/G|_{L(\mathbb{R})}$.

It should be noted here that this may be viewed as a generalization of 0.3, since (I) is equivalent to the existence of some $\theta \in L(\mathbb{R})$,

$$\theta\colon X \to \mathbb{R}$$

such that for all $x_1, x_2 \in A$

$$\exists g \in G(g \cdot x_1 = x_2) \Leftrightarrow \theta(x_1) = \theta(x_2),$$

while (II) is analogous to (and actually by 3.2, 2.7 equivalent with) the existence of $\theta \in L(\mathbb{R})$,

$$\theta\colon \mathbb{R} \to X$$

such that for all $r_1, r_2 \in \mathbb{R}$

$$r_1 - r_2 \in \mathbb{Q} \Leftrightarrow \exists g \in G(g \cdot \theta(r_1) = \theta(r_2)).$$

By the same method one obtains that the orbit structure of a complete left invariant metrizable group never reduces the equality relation on countable sets of reals.

0.5 THEOREM: *Let G be a Polish group with a left invariant complete metric acting continuously on a Polish space X. Then there is no Borel $\theta\colon \mathbb{R}^{\mathbb{N}} \to X$ such that for all $x, y \in \mathbb{R}^{\mathbb{N}}$*

$$\{x(n)\colon n \in \mathbb{N}\} = \{y(n)\colon n \in \mathbb{N}\} \Leftrightarrow \exists g \in G(g \cdot x = y).$$

In the $\mathrm{AD}^{L(\mathbb{R})}$ context this yields that $|\mathcal{P}_{\aleph_0}(\mathbb{R})|_{L(\mathbb{R})} \not\leq |X/G|_{L(\mathbb{R})}$—the effective cardinality of the set of all countable sets of reals is not below that of the set of G-orbits.

Finally, since Becker's result implies Vaught's conjecture for Polish groups admitting a left invariant complete metric, he was led to ask whether these are the **only** Polish groups satisfying Vaught's conjecture. In answer:

0.6 THEOREM: *There is a Polish group G with no compatible left invariant metric such that whenever it acts continuously on a Polish space X, either*

(I) $|X/G| \leq \aleph_0$,

or

(II) $2^{\aleph_0} \leq |X/G|$.

The group arises as $\mathrm{Aut}(M)$, for M a countable model constructed by Julia Knight.

It might be felt that the most important results in this paper are for $\underset{\sim}{\Sigma}^1_1$. Without contesting this I will comment that other kinds of sets do arise naturally in mathematical practice, and a variety of examples are presented in [20]. For instance, the collection of (codes for) continuous functions on Polish space is $\underset{\sim}{\Pi}^1_1$, and hence in $L(\mathbb{R})$, but not $\underset{\sim}{\Sigma}^1_1$. The class of differentiable functions in $C([0,1])$

(continuous real valued functions on the unit interval) is $\underset{\sim}{\Pi}^1_1$ non-$\underset{\sim}{\Sigma}^1_1$, while the class of functions in $C([0,1])$ satisfying the mean value theorem is strictly more complicated than $\underset{\sim}{\Sigma}^1_1$ whilst being easily calculable in $L(\mathbb{R})$. Likewise, in the Borel structure generated by first order logic, the set of rigid linear orders of $\mathbb{N}$ is again in $L(\mathbb{R})$ without being $\underset{\sim}{\Sigma}^1_1$.

However, from the perspective of this work the dividing line between say $\underset{\sim}{\Sigma}^1_1$ and non-analytic but in $L(\mathbb{R})$ is a secondary issue. The real point is that $L(\mathbb{R})$ provides an extreme perspective, containing virtually everything we might allow as reasonably definable; closed under basic set theoretical operations it is rich enough to perform most mathematical activity, and by satisfying DC, the axiom of dependent choice, it remembers sufficient choice for us to reconstruct essentially all classical analysis.

The different sections can be read independently, with only the proofs of §3 requiring a knowledge of determinacy. The background material is spread through §1, §2 and §4, with §5 and §6 requiring §1 and §4, §3 assuming §2 and §1.

1. On Polish groups

This section collects together some background on Polish group actions. Further discussion, along with a few of the proofs and most of the references, can be found in [3], [20] or [21].

1.1 Definition: A topological group is said to be **Polish** if it is Polish as a topological space—which is to say that it is separable and allows a complete metric. If G is a Polish group and X is a Polish space on which it acts continuously, then X is said to be a **Polish G-space**. E^X_G is the orbit equivalence on X, given by

$$x_1 E^X_G x_2 \Leftrightarrow \exists g \in G(g \cdot x_1 = x_2).$$

The orbit $G \cdot x$ of a point x in X is denoted by $[x]_G$. X/G denotes the collection of orbits, $\{[x]_G : x \in X\}$.

1.2 Example: Let S_∞ be the group of all permutations of the natural numbers, and let $2^{\mathbb{N}\times\mathbb{N}}$ be the space of all functions from $\mathbb{N} \times \mathbb{N}$ to $\{0,1\}$. Equip $2^{\mathbb{N}\times\mathbb{N}}$ with the product topology and S_∞ with the topology of pointwise convergence, under which we have that S_∞ is a Polish group and $2^{\mathbb{N}\times\mathbb{N}}$ is a Polish S_∞-space in the action defined by

$$(g \cdot x)(n,m) = x(g^{-1}(n), g^{-1}(m))$$

for any $x \in 2^{\mathbb{N}\times\mathbb{N}}$ and $g \in S_\infty$.

There is a natural sense in which we may view elements of $2^{\mathbb{N}\times\mathbb{N}}$ as coding countable structures whose underlying set is $\mathbb{N}$ and whose only relation is a single binary relation, the extension of which equals $\{(m,n): x(m,n) = 1\}$. If for $x \in 2^{\mathbb{N}\times\mathbb{N}}$ we let $\mathcal{M}_x$ be the corresponding model, then we obtain that for all x_1, x_2 in the space

$$\mathcal{M}_{x_1} \cong \mathcal{M}_{x_2} \Leftrightarrow \exists g \in S_\infty(g \cdot x_1 = x_2).$$

This can be extended in a simple minded fashion to allow elements of $2^{(\Pi_{n\in\mathbb{N}}\mathbb{N}^n)}$ to code models of an arbitrary countable language, and to let S_∞ act so that it again induces isomorphism as its orbit equivalence relation. ∎

In analyzing S_∞ it is often possible to use model theoretic ideas, such as types; in the context of arbitrary Polish group actions we can hope instead to use the notion of **Vaught transforms.**

1.3 Definition: Let G be a Polish group and X a Polish G-space. Then for $B \subset X$, $U \subset G$ open, $B^{\Delta U}$ is the set of $x \in X$ such that for a non-meager set of $g \in U$,

$$g \cdot x \in B;$$

B^{*U} is the set of $x \in X$ such that for a comeager set of $g \in U$,

$$g \cdot x \in B.$$

For $x \in X$, $B \subset X$, $U \subset G$ open,

$$\forall^* g \in U (g \cdot x \in B)$$

indicates that for a relatively comeager set of $g \in U$, $g \cdot x \in B$. Finally,

$$\exists^* g \in U (g \cdot x \in B)$$

is used to indicate that for a non-meager set of $g \in U$, $g \cdot x \in B$. ∎

It is generally only advisable to consider the Vaught transform of B when it is sufficiently well behaved to guarantee that the transforms have the Baire property—for instance, if B is Borel, or in $L(\mathbb{R})$ under suitable hypotheses. In the case that, say, (B_i) is a sequence of Borel sets

$$(\bigcup B_i)^{\Delta U} = \bigcup\{(B_i)^{*V}: V \subset U, V \neq \emptyset, i \in \mathbb{N}\},$$

and thus we obtain that the Vaught transform of a Borel set is again Borel.

For general equivalence relations, induced by a group action, or arising in some other manner, there is a spectrum of ways in which they may be compared, of which I mention those that will be most important in the remainder of the paper.

1.4 Definition: For E and F equivalence relations on Polish spaces $X$and Y, $E \leq_B F$, E **Borel reduces** F indicates that there is a Borel function θ: $X \to Y$ such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2);$$

we write $E \leq_c F$, $E \leq_{\underset{\sim}{\Delta}^1_2} F$, $E \leq_{L(\mathbb{R})} F$, $E \leq_{a\underset{\sim}{\Delta}^1_2} F$, to indicate that there is, respectively, a continuous, $\underset{\sim}{\Delta}^1_2$, $L(\mathbb{R})$, or *absolutely* $\underset{\sim}{\Delta}^1_2$-measurable θ: $X \to Y$ such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2),$$

where a set is said to be **absolutely** $\underset{\sim}{\Delta}^1_2$ if it is defined by two $\underset{\sim}{\Sigma}^1_2$ formulas that continue to define exact complements through all generic extensions; the importance of absolutely $\underset{\sim}{\Delta}^1_2$ is that is provides the most general class of functions for which we maintain reasonable control in ZFC. Here we may assume without loss of generality that X and Y are in $L(\mathbb{R})$, the smallest class inner model of ZF containing the reals. One writes $E \sqsubseteq_B F$, $E \sqsubseteq_c F$ and $E \sqsubseteq_{L(\mathbb{R})} F$ if there is a one-to-one θ that performs the above described reduction, and is Borel, continuous or in $L(\mathbb{R})$, respectively. These notions are graded, since all continuous functions are Borel, all Borel are $\underset{\sim}{\Delta}^1_2$, and all these in turn lie inside $L(\mathbb{R})$. ∎

In this paper I will only be interested in the reductions above. These suggest a notion of bi-reducibility among equivalence relations, defined to hold when

$$E \leq_B F \leq_B E.$$

We might also define a rival notion of equivalence to hold when there is a Borel bijection θ between the underlying Borel spaces X and Y with

$$\forall x_1, x_2 \in X (x_1 E x_2 \Leftrightarrow \theta(x_1) F \theta(x_2)),$$

but it turns out that the definition at 1.4 above better reflects the idea of **effective cardinality**.

1.5 Definition: E_0 is the equivalence relation of eventual agreement on infinite sequences of 0's and 1's, so that for $x, y \in 2^{\mathbb{N}}$

$$x E_0 y \Leftrightarrow \exists N \forall n > N (x(n) = y(n)).$$

It is known that under the ordering of Borel reducibility, E_0 is equivalent to the more familiar Vitali equivalence relation given by

$$x E_v y \Leftrightarrow (x - y) \in \mathbb{Q},$$

in as much as $E_0 \leq_B E_v \leq_B E_0$.

For the first inequality it suffices to consider $(0,1)/\mathbb{Q}$. For $x \in (0,1)$ let $\theta(x) = (r_n(x))_{n \in \mathbb{N}}$ denote the decimal expansion of x with respect to the varying basis $(n)_{n \in \mathbb{N}}$—so that

$$\sum \{r_n(x)/n!: n \in \mathbb{N}\} = x$$

and each $r_n(x) \in \{0, 1, 2, ..., n-1\}$; in the case of there being more than one such expansion—which corresponds to a recurring 9 in an infinite decimal expansion—we can convene to choose the expansion that terminates with $r_n(x) = 0$ for all sufficiently large n. Since the resulting expansion of any rational number has finite support, in the sense of simply being zero everywhere from some point on, we obtain that $x_1 - x_2 \in \mathbb{Q}$ if and only if $\theta(x_1)$ and $\theta(x_2)$ eventually agree. From here we can organize a coding by elements in 2^ω, with similar properties and hence a reduction to E_0. (I am very grateful to Itay Neeman for pointing out this short proof.)

For the second inequality, let $(q_n)_{n \in \omega}$ list the rationals, and choose a family $(V_s)_{s \in 2^{<\omega}}$ of non-empty open sets such that

$$s \subset t \Rightarrow V_s \supset \overline{V}_t,$$
$$s(n) \neq t(n) \Rightarrow \forall m \leq n (q_m \cdot V_s \cap V_t = 0),$$

and for $lh(s) = n$ and $w \in 2^{<\omega}$ we have

$$q_{k(n)} \cdot V_{s0w} = V_{s1w}$$

for some subsequence $(q_{k(n)})_{n \in \omega} \subset (q_n)_{n \in \omega}$, where siw refers to the concatenation of s followed by i and then w. The function θ with

$$\{\theta(x)\} =_{df} \bigcap V_{x|n}$$

for $x \in 2^\omega$ provides the reduction.

While from the point of view of ZFC cardinals, $2^{\mathbb{N}}/E_0$ (or $\mathbb{R}/E_v$) both have cardinality $2^{\aleph_0}$, and hence the same size as $2^{\mathbb{N}}$, or $\mathbb{R}$, from the point of view of **effective cardinals**, these sets are very different. For instance, in $L(\mathbb{R})$ there is no injection from $2^{\mathbb{N}}/E_0$ to $2^{\mathbb{N}}$. Similarly from the context of Borel structure, there is no Borel θ: $2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for all x_1, x_2

$$x_1 E_0 x_2 \Leftrightarrow \theta(x_1) = \theta(x_2).$$

Here $\mathrm{id}(2^\omega)$ is the equality relation on 2^ω, which, as the collection of sequences from $\{0,1\}$, may be identified with $2^{\mathbb{N}}$. $\mathrm{id}(2^{<\omega_1})$ is the equality relation on countable transfinite sequences of 0's and 1's. Again, while 2^ω and $2^{<\omega_1}$ have the same cardinality in ZFC, there is no **reasonably definable** injection from $2^{<\omega_1}$ to 2^ω; under suitable large cardinal assumptions, we have, for instance, no such injection in $L(\mathbb{R})$.

1.6 Definition: HC denotes the collection of all sets whose transitive closure is countable–in other words, if $x \in$ HC then every $x_0 \in x$ is countable, every $x_1 \in x_0 \in x$ is countable, and so on. ∎

It is known from the Scott analysis of [22] and the more recent results of [3] that if X is a Polish S_∞-space then $E^X_{S_\infty} \leq_{a\underset{\sim}{\Delta}^1_2} \mathrm{id}(\mathrm{HC})$, in the sense of there being an absolutely $\underset{\sim}{\Delta}^1_2$ **in the codes** function; that is to say, there is an absolutely $\underset{\sim}{\Delta}^1_2$ function from X to elements of $2^{\mathbb{N}\times\mathbb{N}}$, such that any two $x_1, x_2 \in X$ are orbit equivalent if and only $\theta(x_1)$ and $\theta(x_2)$ code the same element in HC. (Here we can say that $x \in 2^{\mathbb{N}\times\mathbb{N}}$ **codes** $a \in$HC if $(\mathrm{TC}(a)\cup\{a\}, \in) \cong \mathcal{M}_x$, where $\mathcal{M}_x$ is as in 1.2.)

1.7 Definition: A Polish group G is said to be a **cli group** if it has a compatible left invariant complete metric—that is to say there is a compatible complete metric d such that for all $g, h_1, h_2 \in G$

$$d(h_1, h_2) = d(gh_1, gh_2).$$

It is known that all Polish groups have a compatible left invariant metric, but not all have a **complete** left invariant metric. For instance, neither S_∞ nor the homeomorphism group of the unit interval are cli groups. On the other hand, all abelian and locally compact groups are cli groups (see [1]). A group has a left invariant complete metric if and only if it has a right invariant complete metric, since we can pass from one to the other by setting $d^*(g,h) = d(g^{-1}, h^{-1})$.

For left invariant metrics the notion convergence is topological: $(g_i)_{i\in\mathbb{N}}$ will be Cauchy if and only if for each open neighbourhood U of the identity there is some N such that for all $n, m \geq N$, $g_n^{-1}g_m \in U$. Thus, in particular, if one left invariant metric is complete they all are.

The following important fact appears in [26]:

1.8 Lemma: *Let G be a Polish group and X a Polish G-space. Then for $x \in X$ we let the stabilizer of x, $\{g \in G\colon g\cdot x = x\}$, be denoted by G_x. Then $[x]_G$ is uniformly Borel in x and any real coding G_x.*

In particular, every equivalence class is Borel.

Finally:

1.9 THEOREM (Effros): *With X and G as in 1.8, $x \in X$, $[x]_G \in \underset{\sim}{\Pi}^0_2$ if and only if the map $G \to [x]_G$, $g \mapsto g \cdot x$ is open.*

For the purpose of indicating how 1.9 is currently being used we have

1.10 LEMMA: *Let G be a Polish group and X a Polish G-space with basis $\mathcal{B}$, $x \in X$. Let d_G be a compatible metric on G. Suppose that $[x]_G$ is G_δ.*

Then for all $\epsilon > 0$ we may find some $U_\epsilon \in \mathcal{B}$ containing x such that for all $x' \in U_\epsilon \cap [x]_G$ and $U \in \mathcal{B}$,

$$[x]_G \cap U_\epsilon \cap U \neq \emptyset$$

implies that there exists $g \in G$ such that

$$d_G(1, g) < \epsilon \quad \text{and} \quad g \cdot x' \in U.$$

Proof: Let $V \subset G$ be an open neighbourhood of the identity such that $V^{-1} = V$ and $V^2 \subset \{g \in G : d_G(1, g) < \epsilon\}$. Since $g \mapsto g \cdot x$ is an open map by 1.9 we may find $U_\epsilon \in \mathcal{B}$ containing x such that $V \cdot x \supset U_\epsilon \cap [x]_G$. ∎

1.11 THEOREM (Hjorth–Kechris): *Let X be a Polish space and E a $\underset{\sim}{\Sigma}^1_1$ equivalence relation on X with every equivalence class Borel. Then either*

(I) $E_0 \sqsubseteq_c E$,

or

(II) $E \leq_{a\underset{\sim}{\Delta}^1_2} \mathrm{id}(2^{<\omega_1})$ *(i.e. there is an $a\underset{\sim}{\Delta}^1_2$-measurable in the codes reduction).*

Proof: See [17]. ∎

1.12 COROLLARY: *Let A be a $\underset{\sim}{\Sigma}^1_1$ set and E a $\underset{\sim}{\Sigma}^1_1$ equivalence relation on A with every equivalence class Borel. Then either*

(I) $E_0 \sqsubseteq_c E$,

or

(II) $E \leq_{a\underset{\sim}{\Delta}^1_2} \mathrm{id}(2^{<\omega_1})$.

Proof: Fix a Polish space X so that A is the image under of X under some continuous function $\pi: X \twoheadrightarrow A$. Define F to be pullback of E under π—so that $x_1 F x_2$ if and only if $\pi(x_1) E \pi(x_2)$. If $E_0 \sqsubseteq_c F$ then by composing with π we at once have $E_0 \leq_c E$ and so will be finished after observing:

CLAIM: *If $E_0 \leq_c E$ then $E_0 \sqsubseteq_c E$.*

Proof of Claim (this is a well-known and entirely general fact): Let

$$f: 2^\omega \to X$$

witness $E_0 \leq_c E$ and let $P = f[2^\omega]$ be the image of 2^ω under f. Since the preimage of any point is countable under f, we may find Borel $g\colon P \to 2^\omega$ such that $f \circ g(x) = x$ for any $x \in P$. Thus $E|_P$ is Borel and non-reducible to $\mathrm{id}(2^\omega)$, and so the conclusion follows by [11]. (Claim ∎)

So instead suppose that we are in case (II) of 1.11, and $\theta\colon X \to 2^{<\omega_1}$ is an $a\underset{\sim}{\Delta}^1_2$-measurable in the codes witness to $E \leq_{a\underset{\sim}{\Delta}^1_2} \mathrm{id}(2^{<\omega_1})$. By Jankov–von Neumann uniformization (see 18.A [20]) we may find a C-measurable, and hence $a\underset{\sim}{\Delta}^1_2$-measurable, function $\sigma\colon A \to X$ so that for all $a \in A$

$$\pi \circ \sigma(x) = x.$$

Then $\theta \circ \sigma$ puts us in case (II) for E.

2. In $L(\mathbb{R})$

We will need much the same technology as employed in [12], but working with arbitrary Polish spaces. Here I will assume that given a point x in some Polish space X the reader is willing to allow that we can make sense of constructing from x and forming the smallest inner class model, $L[x]$, containing x. Strictly speaking we need instead to fix a real z coding a presentation of X, and express ourselves in terms of constructing from the pair $(z, y(x))$, where $y(x)$ is an element of 2^ω that codes x relative to the presentation coded by z.

Just as an aside, let me briefly indicate how this might be completed rigorously. For X a Polish space we must first choose a complete compatible metric d_X and a countable dense subset $\{a_i : i \in \mathbb{N}\} \subset X$. We then fix recursive bijections $\pi_4\colon \omega^4 \to \omega$, $\pi_3\colon \omega^3 \to \omega$ and define $z \in 2^\omega$ by the specification that

$$z(\pi_4(n,m,i,j)) = 1$$

if and only if

$$\frac{n}{2^m} < d_X(x_i, x_j).$$

Clearly from z we may recover $d_X(x_i, x_j)$ by considering the cut obtained in the dyadic rationals. From the set $\{d_X(x_i, x_j)\colon i, j \in \mathbb{N}\}$ we can reconstruct the behavior of the complete metric space on a countable dense set, and thus determine it up to isometry. For any $x \in X$ we may define a code $y(x) \in 2^\omega$ by

$$(y(x))(\pi_3(n,m,i)) = 1$$

if and only if

$$\frac{n}{2^m} < d_X(x, x_i).$$

Alternatively, the reader may interpret the results below as holding only for the recursive Polish spaces, in the sense of [25], but allowing the usual relativizations to a parameter. Finally, we may use the fact that all uncountable Polish spaces are Borel isomorphic to code any Polish space by elements of $2^{\mathbb{N}}$, again with the coding taking place relative to some parameter z.

The truth is that any mathematicians working in this field will have their own methods of coding, and so instead of being precise and strict, I will be more informal and treat the elements of any Polish space in exactly the same fashion as the **recursive Polish spaces**, such as $\mathbb{R}$ and 2^ω. Let us agree to only keep in the background that this is not quite accurate but more concise and easily rectifiable.

However the reader chooses to explicate the notion of constructing from a point in a Polish space or using a parameter to code such a space, the notation $x \in X \cap M$, for M an inner model, means that x is a point coded by a real in M, and that the parameter used to code X exists in M. We can think of U as being an open set coded in M if there are sequences $(q_i)_{i\in\mathbb{N}}$ of rationals $(x_i)_{i\in\mathbb{N}}$ of points in X, both in M, such that U equals the set of elements $x \in X$ for which there is some i with x within distance q_i of x_i.

Up to isomorphism, all Polish spaces exist in $L(\mathbb{R})$. Thus it will be convenient to have a standing assumption that all our Polish spaces are in fact elements of this inner model; the assumption can be made without loss of generality.

The theory of $L(\mathbb{R})$ will be developed under the determinacy assumption $\mathrm{AD}^{L(\mathbb{R})}$, which states that every subset of ω^ω in $L(\mathbb{R})$ is determined—one of the players has a winning strategy in the infinite game where I and II alternate in playing integers, and the victor is decided on the basis of whether the resulting element of ω^ω is in the specified subset. While ZFC alone is too weak to decide many natural questions regarding $L(\mathbb{R})$, the assumption of $\mathrm{AD}^{L(\mathbb{R})}$ provides a canonical theory for this inner model. There is widespread acceptance of this assumption in the study of $L(\mathbb{R})$ among set theorists—partly because it leads to a theory for the sets of reals in $L(\mathbb{R})$ which continues the pattern we find for the Borel sets under ZFC, and partly because $\mathrm{AD}^{L(\mathbb{R})}$ was shown in [30] to follow from large cardinal assumptions, such as the existence of a supercompact.

2.1 Definition: If X is a Polish space, $A \subset X$ is said to be ∞-Borel if there is an ordinal α, a set $S \subset \alpha$ and a formula φ such that A equals

$$\{x \in X \colon L_\alpha[x, S] \models \varphi(x, S)\}.$$

2.2 THEOREM (Woodin): *Assume* $\mathrm{AD}^{L(\mathbb{R})}$ *and let* $A \subset X$ *be* $\mathrm{OD}_x^{L(\mathbb{R})}$ *for some real* $x \in \mathbb{R}$. *Then there is an* ∞*-Borel code for* A *in* $\mathrm{HOD}_x^{L(\mathbb{R})}$. (See [27].)

2.3 Definition: Let C be a transitive set in M, a class inner model. Then OD_C^M is the class of all sets Σ_2 definable in the Levy hierarchy from parameters in the ordinals and $\{C\} \cup C$ (as calculated from the point of view of M); HOD_C^M is the class of all sets in OD_C^M whose transitive closure is again included in OD_C^M. It is a standard fact that HOD_C^M is the smallest transitive inner model containing C, the ordinals, and closed under ordinal definability from the point of view of M. For X a Polish space whose presentation exists in M, we use $\mathbb{B}(C, X, M)$ to denote $\{A \in (\mathcal{P}(X))^M \colon A \in \mathrm{OD}_C^M\}$; this can be viewed as a Boolean algebra in the natural sense. (Note that $(\mathcal{P}(X))^M$ refers not to the true power set of X, but the power set of $X \cap M$ inside M.) For $G \subset \mathbb{B}(C, X, M)$ a sufficiently generic filter, we may define a point $x(G) \in X$ by the requirement that for all open $U \subset X$ whose code exists in M we have that

$$x \in U \Leftrightarrow (U \cap M) \in G.$$

The statement of the next theorem is slightly more general than is usual; the proof however follows exactly as does the usual proof, given in [12]. The one variation is that here our inner model HOD_C^M need not satisfy choice.

2.4 THEOREM (Vopenka): *Fix* M, C *and* X *as above, and assume that* C *includes a code for* X. *Then there exists* $\mathbb{B}$ *in* HOD_C^M, $i : \mathbb{B} \cong \mathbb{B}(C, X, M)$ *in* M*, such that:*

(i) *for all* $x \in X \cap M$,

$$G(x) =_{df} \{i^{-1}(A) \colon x \in A, A \in \mathrm{OD}_C^M\}$$

is HOD_C^M*-generic for* $\mathbb{B}$;

(ii) *there is a* HOD_C^M*-generic for* $\mathbb{B}$ *below every non-zero element in* M;

(iii) *if* $H \subset \mathbb{B}$ *is* HOD_C^M*-generic, if we let* $G = i[H]$ *then* $x(G) \in \mathrm{HOD}_C^M[H]$, *and for all ordinals* α, $\vec{c} \in C^{<\omega}$, *and formulas* φ,

$$L_\alpha(C, x(G)) \models \varphi(\vec{c}, x(G)) \Leftrightarrow \{x \in X \cap M \colon M \models \varphi(\vec{c}, x)\} \in G;$$

(iv) $i \in OD_C^M$.

Note that $\mathbb{B}$ from this theorem will have size at most $\mathcal{P}(X)^M$—the set of all subsets of X in M—and so will have cardinality at most $(2^{2^{\aleph_0}})^M$.

The following important result may be found in [24]:

2.5 THEOREM (Martin–Moschovakis–Steel): *Assume* $AD^{L(\mathbb{R})}$. *Then in* $L(\mathbb{R})$, Scale($\underset{\sim}{\Sigma}^2_1$).

It follows from entirely general facts that every non-empty $\underset{\sim}{\Sigma}^2_1$ collection of sets of reals in $L(\mathbb{R})$ has a $\underset{\sim}{\Sigma}^2_1$ member, and thus there will be a member of this collection which is the projection of a **tree**, in the sense of [25].

2.6 Definition: For $A \in L(\mathbb{R})$, $|A|_{L(\mathbb{R})}$ denotes the cardinality of A as calculated in $L(\mathbb{R})$. So $|A|_{L(\mathbb{R})} \leq |B|_{L(\mathbb{R})}$ if there is an injection from A to B; by Schroeder–Bernstein, they have the same cardinality only if there is a bijection between them. For κ an ordinal, $\mathrm{H}(\kappa)$ denotes the collection of sets whose transitive closure has size less than κ. Thus the class of all hereditarily wellorderable sets in $L(\mathbb{R})$ is the union $\bigcup_{\kappa \in \mathrm{Ord}}(\mathrm{H}(\kappa))^{L(\mathbb{R})}$. HC equals $\mathrm{H}(\omega_1)$. ∎

Here it is worth collecting together some facts about $L(\mathbb{R})$-cardinals under the assumption of $AD^{L(\mathbb{R})}$. Note that $H(\omega_1) = H(\omega_1)^{L(\mathbb{R})}$.

2.7 THEOREM (folklore): *Assume* $AD^{L(\mathbb{R})}$. *Then*

(i) $|\mathbb{R}|_{L(\mathbb{R})} \leq |2^\omega|_{L(\mathbb{R})} \leq |\mathbb{R}|_{L(\mathbb{R})}$;
(ii) $|\mathbb{R}|_{L(\mathbb{R})} \not\leq |\omega_1|_{L(\mathbb{R})} \not\leq |\mathbb{R}|_{L(\mathbb{R})}$;
(iii) $|\mathbb{R}|_{L(\mathbb{R})} < |\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})}$;
(iv) $|\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})} \not\leq |(2^\alpha)^{L(\mathbb{R})}|_{L(\mathbb{R})}$ *for any ordinal* α;
(v) $|\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})} \leq |2^\omega/E_0|_{L(\mathbb{R})} \leq |\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})}$;
(vi) $|\omega_1|_{L(\mathbb{R})} < |2^{<\omega_1}|_{L(\mathbb{R})} < |H(\omega_1)|_{L(\mathbb{R})}$;
(vii) $|2^\omega|_{L(\mathbb{R})} < |2^{<\omega_1}|_{L(\mathbb{R})}$.

Proofs: (i) This is clear even without any sort of determinacy assumptions, since there are Borel injections both ways.

(ii) As can be found in [18], there is a countably complete ultrafilter on ω_1 under AD, so there can be no ω_1 sequence of reals in $L(\mathbb{R})$.

(iii) $|\mathbb{R}|_{L(\mathbb{R})} \leq |\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})}$ since we can find a map from $\mathbb{R}$ to $\mathbb{R}$ such that any two distinct reals have images that are mutually generic over $L_{\omega_1^{ck}}$. To see the failure of reducibility in the other direction, note that by the Lebesgue density theorem any $\mathbb{Q}$-invariant Lebesgue measurable function from $\mathbb{R}$ must be constant almost everywhere; since all functions are Lebesgue measurable in $L(\mathbb{R})$, this suffices.

(iv) Let $\theta\colon \mathbb{R} \to 2^\alpha$ be $\mathbb{Q}$-invariant and in $L(\mathbb{R})$. Then, as in the proof of (iii), for each β less than α, the set $\{x \in \mathbb{R} : \theta(x) = 1\}$ is either null or co-null. By Fubini's theorem in $L(\mathbb{R})$ and all sets Lebesgue measurable, wellordered intersections of co-null sets are co-null, and so θ must be constant almost everywhere.

(v) This follows as in the remarks after 1.5, since we have $E_v \leq_B E_0 \cdot \leq_B E_v$.

(vi) Any countable ordinal α can be coded by a function in $2^{<\omega_1}$ that has domain α and takes constant value 1. The other inequality follows since $2^{<\omega_1} \subset H(\omega_1)$. The first *strict* inequality, $|\omega_1|_{L(\mathbb{R})} < |2^{<\omega_1}|_{L(\mathbb{R})}$, follows by $|\mathbb{R}|_{L(\mathbb{R})} \leq 2^{<\omega_1}$ and (ii); the second strict inequality by $|\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})} \leq |H(\omega_1)|_{L(\mathbb{R})}$.

(vii) The non-reduction follows since there is no ω_1 sequence of reals. ∎

A number of results similar to (iii), (vi) and (vii) are presented in [5].

2.8 THEOREM (Woodin): *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *If $E \in L(\mathbb{R})$ on $\mathbb{R}$, then exactly one of the following holds:*

(I) $\mathrm{id}(\mathbb{R}) \sqsubseteq_c E$,

or

(II) *for some ordinal* κ,

$$E \leq_{L(\mathbb{R})} \mathrm{id}(\kappa).$$

2.9 COROLLARY TO THE PROOF: *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Then for any set A, exactly one of the following holds:*

(I) $|\mathbb{R}|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$,

or

(II) *for some ordinal* κ,

$$|A|_{L(\mathbb{R})} \leq |\kappa|_{L(\mathbb{R})}.$$

2.10 THEOREM (Hjorth): *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *If $E \in L(\mathbb{R})$ on $\mathbb{R}$, then exactly one of the following holds:*

(I) $E_0 \sqsubseteq_c E$,

or

(II) *for some ordinal* κ,

$$E \leq_{L(\mathbb{R})} \mathrm{id}(2^\kappa).$$

(See [12].)

2.11 COROLLARY TO THE PROOF: *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Then for any set A, exactly one of the following holds:*

(I) $|\mathbb{R}/\mathbb{Q}|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$,

or

(II) *for some ordinal* κ,

$$|A|_{L(\mathbb{R})} \leq |2^\kappa|_{L(\mathbb{R})}.$$

Theorem 2.10 follows by arguments similar to those used in proving 2.8. It is unknown whether there is an analogue of these results for the cardinality of HC in $L(\mathbb{R})$, but it is known that any such result would need to be considerably more complex—in particular, there is no **single** set B in $L(\mathbb{R})$ so that

$$|A|_{L(\mathbb{R})} \leq |H(\kappa)|_{L(\mathbb{R})}$$

for some ordinal κ if and only if

$$|B|_{L(\mathbb{R})} \leq |A|_{L(\mathbb{R})}$$

fails.

2.12 LEMMA: *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* E *and* F *be Borel—or even* $\underset{\sim}{\Delta}^2_1$, *or even projective—equivalence relations on Polish spaces* X *and* Y. *Then* $E \leq_{L(\mathbb{R})} F$ *if and only if* $|X/E|_{L(\mathbb{R})} \leq |Y/F|_{L(\mathbb{R})}$.

Proof: The *only if* direction is immediate, so suppose that

$$|X/E|_{L(\mathbb{R})} \leq |Y/F|_{L(\mathbb{R})}.$$

Then we can find a set $R \subset X \times Y$ in $L(\mathbb{R})$ so that:

(i) $\forall (x_1, y_1), (x_2, y_2) \in R$, $x_1 E x_2$ if and only if $y_1 F y_2$;

(ii) $\forall x \in X \exists y \in Y((x,y) \in R)$.

Thus by 2.5 we can find such a set R with $R \in \underset{\sim}{\Sigma}^2_1$, and then a tree T on some ordinal with $p[T] = R$. Then by the absoluteness of illfoundedness for trees, we can find in each model $L[T,x]$ some $y \in Y$ with

$$(x,y) \in p[T].$$

Note here that we can define from x and T a wellorder of $L[T,x]$. Thus we may define θ: $X \to Y$ by letting $\theta(x)$ be the first y above in the canonical in (x,T) wellorder of $L[T,x]$. ∎

Consequently it is natural to use the ordering $\leq_{L(\mathbb{R})}$—and by analogy $\leq_B$—in comparing Borel equivalence relations, since this is the notion of comparison that corresponds to cardinality in $L(\mathbb{R})$.

On the other hand:

2.13 LEMMA (folklore): *Let* $X \in L(\mathbb{R})$ *be a non-empty set. Then there is a* $\pi \in L(\mathbb{R})$ *and ordinal* α *such that*

(i) π: $\mathbb{R} \times \alpha \to X$ *is onto;*

and thus

(ii) *there is a sequence* $(E_\beta)_{\beta\in\alpha}$ *of equivalence relations in* $L(\mathbb{R})$ *and* $A \subset \{([x]_{E_\beta}, \beta)\colon \beta < \alpha, x \in \mathbb{R}\}$, *and a bijection* $\sigma\colon A \to X$, $A, \sigma \in L(\mathbb{R})$.

And therefore the study of cardinalities in $L(\mathbb{R})$ is largely the study of definable equivalence relations and their corresponding quotient spaces.

The following result, stated in a rather narrow form, places the results from §3 in context.

2.14 THEOREM (Becker–Kechris): *Assume* $\mathrm{AD}^{L(\mathbb{R})}$. *Let* G *be a Polish group acting continuously on a separable metric space* A *in* $L(\mathbb{R})$. *Then* $E_G^A \sqsubseteq_{L(\mathbb{R})} E_G^X$ *for some Polish* G*-space* X. (See [3].)

Thus whenever A is a separable metric space in $L(\mathbb{R})$ and a Polish group G acts on A, then there will be a Polish G-space X and $B \subset X$ in $L(\mathbb{R})$ so that

$$|A/G|_{L(\mathbb{R})} = |B/G|_{L(\mathbb{R})}.$$

To this degree the study of effective cardinalities induced by Polish group actions may be subsumed in the development of Polish groups acting continuously on Polish spaces.

Theorem 2.14 is given at 5.3.4 of [3]; the assumption of $\mathrm{AD}^{L(\mathbb{R})}$ is only needed so that all sets in a Polish space in $L(\mathbb{R})$ have the property of Baire. However, the more recent construction of a universal Polish G-space in [15] is sufficient to prove 2.14 in ZFC.

3. Generalized Ulm-type dichotomies

The next theorem is stated under entirely abstract hypotheses, assuming ZF, DC—the axiom of dependent choice—and some manner of exotic regularity property for the relevant sets of reals. Of course, the main interest is in the consequences for $L(\mathbb{R})$, and the precise statement below is of technical interest.

As for a word regarding the coding, we may speak of a suitable point $x \in 2^{\mathbb{N}\times\mathbb{N}}$ as coding a transitive set N_x which arises from the Mostowski collapse,

$$\pi_x\colon \omega \twoheadrightarrow N_x,$$

so that for each $n, m \in \omega$ we have

$$x(n,m) = 1 \Leftrightarrow \pi_x(n) \in \pi_x(m).$$

Then I want to take the further step of using subsets of ω to code generic filters over N_x, so that if P is a partial order and $(p_i)_{i\in\omega}$ is some enumeration then we may say that $y \in 2^\omega$ codes a filter G_y on P (relative to the enumeration $p_0, p_1, \ldots$) if

$$G_y =_{df} \{p_n : y(n) = 1\}$$

is a filter on P. Given a countable structure N satisfying some minimal amount of set theory, a partial order $P \in N$ and a filter $G \subset P$, we can form the generic extension $N[G]$ of N by closing under the terms in N. In the usual manner we can say that a collection of filters $A \subset 2^P$ is Borel in the codes or $\underset{\sim}{\Sigma}^1_1$ in the codes if it is uniformly Borel or $\underset{\sim}{\Sigma}^1_1$ in any enumeration of P.

3.1 LEMMA: *Let M be a countable transitive model, $\vec{c} \in M$ a finite tuple, $P \in M$ a partial order and φ a formula in set theory. Then*

$$\{G \in 2^P : M[G] \models \varphi(\vec{c}, G)\}$$

is uniformly $\Delta^1_1(w)$ for any parameter w coding M, the tuple $\vec{c}$ and the partial order P.

Proof: Recall the standard fact that it is uniformly $\Delta^1_1(x)$ to calculate the theory of the model, M_x, coded by x. Now the lemma follows since the collection of codes for $M[G_y]$ is uniformly $\Delta^1_1(w, y)$ for any w as above and $y \in 2^\omega$ coding a filter on P. ∎

3.2 THEOREM: *Assume ZF, DC, all sets of reals are ∞-Borel, and that there is no ω_1 sequence of reals. Let H be a Polish group, X be a Polish H-space, and $A \subset X$ an H-invariant subset. Then either:*

(I) $E^X_H|_A \leq \mathrm{id}(2^{<\omega_1})$,

or

(II) $E_0 \sqsubseteq_c E^X_H|A$.

Remark: Here the unadorned $\leq$ means that there just outright exists a reduction θ, with no special definability assumption. In the context of ZF+¬AC this notion has content.

Proof: Let φ, α, and $S \subset \alpha$ witness the definition of ∞-Borel, so that A is equal to the set of $x \in X$ such that

$$L_\alpha[S, x] \models \varphi(S, x).$$

Without loss of generality S codes X, H, and the action, in some appropriate sense.

[The idea of the proof is this: We would dearly like to use the Vopenka algebra relative to S to mimic the usual Glimm–Effros dichotomy theorems with a forcing proof over HOD_S. In general the combinatorics of this may fare poorly, since the forcing involved is extravagant and there is no hope of any kind of absoluteness; and even if the combinatorics succeed the potentially vast size of the collection of OD_S subsets of X would seem to offer little hope to obtain countable objects as complete invariants.

So instead we reflect down to $(\mathrm{HOD}_S)^{L[x,S]}$ for various $x \in A$. There arises a new difficulty since $(\mathrm{HOD}_S)^{L[x,S]}$ may not be an invariant of the orbit $[x]_H$. This further problem can be solved by 'averaging' out along the ideal on the orbit obtained by the ideal of meager sets in the group. So we consider $(\mathrm{HOD}_S)^{L[g\cdot x,S]}$ for 'generic' $g \in H$; this *does* provide an invariant of the orbit. At this stage we focus on the small piece of A that can arise in forcing over $(\mathrm{HOD}_S)^{L[g\cdot x,S]}$ with the Vopenka algebra. If $E_0 \sqsubseteq_c E_H^X|A$ **fails**, then on each of these small pieces we can uniformly assign bounded subsets of ω_1 as complete invariant. The various assignments might overlap, and so as our final invariant we take not just the bounded subset assigned to $[x]_H$ using $(\mathrm{HOD}_S)^{L[g\cdot x,S]}$ but also a code of a large initial segment of $(\mathrm{HOD}_S)^{L[g\cdot x,S]}$; this last code will be uniformly realizable as a bounded subset of ω_1 by the existence of a uniform in S wellorder of $(\mathrm{HOD}_S)^{L[g\cdot x,S]}$.]

Since there is no ω_1-sequence of reals, ω_1 is strongly inaccessible in $L[S,x]$ for any $x \in X$. Thus, in particular, almost every $g \in H$ is generic over $L[S,x]$ for the forcing notion that uses the non-empty basic open sets of H ordered under inclusion as a forcing notion. This notion is equivalent to Cohen forcing and homogeneous; thus as in the standard development of forcing, presented by [18], the corresponding HOD of the generic extension is decided in the ground model, and

$$\forall^* g \in H \forall^* h \in H(\mathrm{HOD}_S^{L[S,g\cdot x]} = \mathrm{HOD}_S^{L[S,h\cdot x]}).$$

Let M_S^x denote this common model, so that

$$\forall^* g \in H(M_S^x = \mathrm{HOD}_S^{L[S,g\cdot x]}).$$

Note then that M_S^x depends only on $[x]_H$. Define the ordinal $\gamma(x)$ to be the first beth fixed point after ω in the resulting model for a comeager set of $g \in H$:

$$|V_{\gamma(x)}|^{L[S,g\cdot x]} =_{df} (\beth_{\gamma(x)})^{L[S,g\cdot x]} = \gamma(x).$$

By strong inaccessibility of ω_1^V this ordinal will be countable. The key facts about M_S^x are that it has a uniformly $\Sigma_2(S)$ wellorder and all the forcing below takes place inside the V-countable structure $(V_{\gamma(x)})^{M_S^x}$.

Then let B_x be the set of $y \in X$ such that for some $h \in H$ and $p \in \mathbb{B}_S^x$ we have:

(a) there is an M_S^x-generic $G \subset \mathbb{B}_S^x$ with

$$x(G) = h \cdot y;$$

(b) p forces that $x(G) \in A$, in the sense that

$$p \Vdash L_\alpha[S, x(\dot{G})] \models \varphi(S, x(\dot{G})),$$

where $\dot{G}$ is understood to be the name for the generic object on $\mathbb{B}_S^x$.

Let p_x be maximal so that

$$p_x \Vdash L_\alpha[S, x(\dot{G})]] \models \varphi(S, x(\dot{G}));$$

this exists by completeness of the algebra $\mathbb{B}_S^x$. (In fact p_x is the image of $A \cap L[S, g \cdot x]$ under the canonical isomorphism of $\mathbb{B}_S^x$ and the algebra of OD subsets of X in $L[S, g \cdot x]$ for any sufficiently generic $g \in H$.)

Claim (1): *B_x is uniformly $\Sigma_1^1(w)$ for any code w for the parameters H, X, p_x, $\gamma(x)$, $(V_{\gamma(x)})^{M_S^x}$, and $\mathbb{B}_S^x$.*

Proof of Claim: Let A_x be the set of triples (y, g, G) so that

$$G \subset \mathbb{B}_S^x$$

is M_S^x generic below p_x and

$$x(G) = g \cdot y.$$

By 3.1 this set is uniformly Borel in the indicated parameters. Since $B_x = \{y : \exists g \in H, G \in 2^{\mathbb{B}_S^x}((y, g, G) \in A_x)\}$ we have the claim for B_x. (Claim (1) ∎)

Note that $E_H^X|B_x$ is uniformly $\utilde{\Sigma}_1^1$ in any code for the parameters indicated above. If for some x we have $E_0 \sqsubseteq_c E_H^X|B_x$ then certainly $E_0 \sqsubseteq_c A$, since $B_x \subset A$, and the proof is finished. So suppose instead that for every $x \in A$ this is not the case, and we apply 1.12 to obtain a reduction to $2^{<\omega_1}$.

Fixing $x \in A$ let $\mathbb{P}_S^x = \mathrm{Coll}(\omega, \gamma(x))$. Note that for any $G' \subset \mathbb{P}_S^x$ that is M_S^x generic we have that there is a code for B_x as a $\utilde{\Sigma}_1^1$ set by Claim (1) above.

Claim (2): $(M_S^x)^{\mathbb{P}_S^x} \models E_H^X|B_x \leq_{a\utilde{\Delta}_2^1} \mathrm{id}(2^{<\omega_1})$.

Proof of Claim: Instead, suppose $q \in \mathbb{P}_S^x$ forces otherwise. Then we may find $G' \subset \mathbb{P}_S^x$ in V that is M_S^x generic below q. By 1.12 we must have

$$E_0 \sqsubseteq_c E_H^X|_{B_x}$$

in $M^x_S[G']$. But now $E_0 \sqsubseteq_c E^X_H|B_x$ is certainly no worse than $\utilde{\Sigma}^1_3$, hence upwards absolute between class models, and thus must hold also in V, with a contradiction to the case assumption. (Claim (2) ∎)

Let $\dot\theta^x_S$ be the term for an $a\utilde{\Delta}^1_2$-measurable function witnessing Claim (2) in $(M^x_S)^{\mathbb{P}^x_S}$. Note by Shoenfield that for any $G' \subset \mathbb{P}^x_S$ that is M^x_S-generic and any $x_1, x_2 \in B_x$, if $\dot\theta^x_S[G'](x_1)$ and $\dot\theta^x_S[G'](x_2)$ are defined then

$$x_1 E^X_H x_2 \Leftrightarrow \dot\theta^x_S[G'](x_1) = \dot\theta^x_S[G'](x_2).$$

Note by the definition of $a\utilde{\Delta}^1_2$ that for any such M^x_S-generic $G' \subset \mathbb{P}^x_S$ the definition of $\dot\theta^x_S[G']$ will continue to describe a reduction of B_x to $\mathrm{id}(2^{<\omega_1})$ through all future generic extensions of $M^x_S[G']$.

For any $y \in B_x$ we now let $\hat\theta^x_S(y)$ be the set of pairs of countable ordinals (α_0, α_1) such that for q the α_0th object in M^x_S in the canonical well order we have

$$q \in \mathbb{P}^x_S$$

and that there exists some M^x_S-generic $G \subset \mathbb{B}^x_S$ and $g \in H$ so that

$$g \cdot y = x(G) \quad \text{and} \quad M^x_S[G] \models q \Vdash \dot\theta^x_S(g \cdot y) = \alpha_1.$$

It follows from the structure of the definition that $\hat\theta^x_S(y)$ is an invariant of $[y]_H$.

Claim (3): *Suppose $x_0, x_1 \in A$ with $\mathbb{B}^{x_0}_S = \mathbb{B}^{x_1}_S$, $\mathbb{P}^{x_0}_S = \mathbb{P}^{x_1}_S$, $\gamma(x_0) = \gamma(x_1)$ and $\dot\theta^{x_0}_S = \dot\theta^{x_1}_S$ (as terms in the forcing notion $\mathbb{B}^{x_0}_S = \mathbb{B}^{x_1}_S$). If $y_0 \in B_{x_0}$ and $y_1 \in B_{x_1}$ are E^X_H-inequivalent, then*

$$\hat\theta^{x_0}_S(y_0) \neq \hat\theta^{x_1}_S(y_1).$$

Proof of Claim: Choose $g_0, g_1 \in H$, $G_0, G_1 \subset \mathbb{B}^{x_0}_S (= \mathbb{B}^{x_1}_S)$ witnessing $g_0^{-1} \cdot x(G_0) = y_0 \in B_{x_0}$ and $g_1^{-1} \cdot x(G_1) = y_1 \in B_{x_1}$, respectively. Choose $G' \subset \mathbb{P}^{x_0}_S = \mathbb{P}^{x_1}_S$ to be both $M^{x_0}_S[G_0]$ and $M^{x_1}_S[G_1]$ generic.

Subclaim: $(\dot\theta^{x_0}_S[G'])(g_0 \cdot y_0) = (\dot\theta^{x_1}_S[G'])(g_0 \cdot y_0) \neq (\dot\theta^x_S[G'])(g_1 \cdot y_1)$.

Proof of Subclaim: By the product lemma of §20 [18] we have that $g_0 \cdot y_0$ and $g_1 \cdot y_1$ are generic over $M^{x_0}_S[G']$ and $M^{x_1}_S[G']$. Thus by the definition of $a\utilde{\Delta}^1_2$ we have that $(\dot\theta^{x_0}_S[G'])(g_0 \cdot y_0)$ and $(\dot\theta^{x_1}_S[G'])(g_1 \cdot y_1) = (\dot\theta^{x_0}_S[G'])(g_1 \cdot y_1)$ are well defined. Now the Subclaim follows by the absoluteness of $\utilde{\Pi}^1_2$ between $M^{x_1}_S[G']$ and V. (Subclaim ∎)

Thus without loss of generality there will be some $\alpha_1 < \omega_1$ so that

$$\alpha_1 \in (\dot\theta^{x_1}_S[G'])(g_0 \cdot y_0) \setminus (\dot\theta^{x_1}_S[G'])(g_1 \cdot y_1).$$

Then for $q \in G'$ so that

$$M_S^{x_0}[G_0] \models q \Vdash \alpha_1 \in \dot{\theta}_S^{x_0}(g_0 \cdot y_0),$$

and α_0 so that q is the α_0th object in $M_S^{x_0}$,

$$(\alpha_0, \alpha_1) \in \hat{\theta}_S^{x_0}(y_0) \setminus \hat{\theta}_S^{x_0}(y_1). \text{ (Claim (3) ∎)}$$

The only problem with this is that the reduction $\hat{\theta}_S^x$ only provides a reduction to $\mathrm{id}(2^{<\omega_1})$ on each slice of the form B_x; potentially we may have $y_0 \in B_{x_0}$ and $y_1 \in B_{x_1}$ with $[x_0]_H \neq [x_1]_H$ and

$$\hat{\theta}_S^{x_0}(y_0) = \hat{\theta}_S^{x_1}(y_1).$$

So using some injection $\langle \cdot, \cdot, \cdot, \ldots \rangle \colon \omega_1^7 \hookrightarrow \omega_1$ we can let $\theta(x)$ for $x \in A$ be the set of

$$\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \gamma(x), \delta_0, \delta_1 \rangle$$

such that

(i) $\alpha_0, \alpha_1 < \omega_1$, $\alpha_2, \alpha_3 < \gamma(x)$;
(ii) the α_2th object in the canonical well order of $(V_{\gamma(x)})^{M_S^x}$ is an element of the α_3th object in the canonical well order of $(V_{\gamma(x)})^{M_S^x}$;
(iii) the δ_0th object in $(V_{\gamma(x)})^{M_S^x}$ is $\mathbb{B}_S^x$;
(iv) the δ_1th object in $(V_{\gamma(x)})^{M_S^x}$ is the term $\dot{\theta}_S^x$;
(v) $(\alpha_0, \alpha_1) \in \hat{\theta}_S^x(x)$.

It should be clear from the structure of the definitions that $\theta(x)$ is an invariant of the orbit $[x]_H$. By (i)–(iv) we have that if $\theta(x_0) = \theta(x_1)$ then

$$\mathbb{B}_S^{x_0} = \mathbb{B}_S^{x_1}, \quad \gamma(x_0) = \gamma(x_1), \quad \dot{\theta}_S^{x_0} = \dot{\theta}_S^{x_1};$$

but then it follows by Claim (3) that $x_0 E_H^X x_1$. ∎

There is a fact implicit in this proof that should be explicitly mentioned: For any Polish group H and Polish H-space X there is in $L(\mathbb{R})$ an H-invariant

$$\rho \colon X \to 2^{<\omega_1}$$

such that for any $Z \subset 2^{<\omega_1}$ both the set

$$\{x \in X \colon \rho(x) = Z\}$$

and the relation

$$E_H|_{\{x \in X : \rho(x) = Z\} \times \{x \in X : \rho(x) = Z\}}$$

are uniformly Borel in any real coding Z.

3.3 COROLLARY: *Assume* $AD^{L(\mathbb{R})}$. *Let* G *be a Polish group and* X *a Polish* G*-space, and let* $A \subset X$ *be in* $L(\mathbb{R})$. *Then either:*

(I) $E_G^X|_A \leq_{L(\mathbb{R})} \mathrm{id}(2^{<\omega_1})$,

or

(II) $E_0 \sqsubseteq_c E_G^X|_A$.

Proof: By 2.2 we have all sets of reals ∞-Borel in $L(\mathbb{R})$. As at 2.7, the determinacy assumption implies there is no ω_1-sequence of reals, and we have the assumptions of 3.2. ∎

In unpublished work Woodin has previously obtained from $AD_{\mathbb{R}}$ that all sets are the projection of some tree on some ordinal, and hence ∞-Borel; thus under ZF+DC+$AD_{\mathbb{R}}$ either $E_G^X|_A \leq \mathrm{id}(2^{<\omega_1})$ or $E \sqsubseteq_c E_G^X|_A$, thereby answering question 8.1.2 from [3].

The method of proof at 3.2 can also be adapted to show that if $E_H^X \leq_{L(\mathbb{R})} \mathrm{id}(H(\kappa))$ for some ordinal κ then the orbit equivalence relation may be reduced to $\mathrm{id}(HC)$. The main point is that we may assume that any such reduction

$$\theta\colon E_H^X \leq H(\kappa)$$

is T-Souslin for T a tree for $\underset{\sim}{\Sigma}_1^2$, and then we may assign to any $x \in X$ some suitably chosen piece of $\mathrm{HOD}_{z,T,\theta(x)}^{L[g\cdot x,z,T,\theta(x)]}$, where $g \in H$ generic and z is a parameter coding the action. More recently, rather different techniques in [16] have shown the stronger result that when E_G^H is $L(\mathbb{R})$ reducible to isomorphism on **wellorderable** models, then it is reducible to isomorphism on countable structures.

4. Infinitary logic and group actions

This section summarizes the main points in the development of infinitary logic for descriptive set theory and ∞-Borel codes. These remarks are along the lines of [8] and [10], but with particular emphasis on the context of Polish group actions.

The unattributed results here are technical and largely folklore. They will form the background for §5.

4.1 LEMMA: *Let* X *be a Polish space,* $\mathcal{B}$ *a basis for the topology, and* $C \subset X$ *a closed subset, then* $\mathcal{B}^C = \{O \cap C : O \in \mathcal{B}\} \cup \mathcal{B}$ *is a basis of a new topology on* X.

Proof: Since X with the new topology is homeomorphic to the disjoint union of C and $X \setminus C$, both of which are shown in [20] to be Polish in the relative topology. ∎

4.2 Definition: Let X be a Polish space and $\mathcal{B}$ a basis. Let $\mathcal{L}(\mathcal{B})$ be the propositional language formed from atomic propositions of the form '$\dot{x} \in U$', for $U \in \mathcal{B}$. Let $\mathcal{L}_{\omega_1 0}(\mathcal{B})$ be the infinitary version, obtained by closing under negation and countable disjunction and conjunction, and let $\mathcal{L}_{\infty 0}(\mathcal{B})$ be the obtained by closing under arbitrary Boolean operations. $F \subset \mathcal{L}_{\infty 0}(\mathcal{B})$ is a **fragment** if it is closed under subformulas and the finitary Boolean operations of negation and finite disjunction and finite conjunction. For $\varphi \in \mathcal{L}_{\infty 0}(\mathcal{B})$, $F(\varphi)$, the **fragment generated by** φ is the smallest fragment containing φ.

For a point $x \in X$ and $\varphi \in \mathcal{L}_{\infty 0}(\mathcal{B})$, we can then define $x \models \varphi$ by induction in the usual fashion: If $\varphi =$ '$\dot{x} \in U$' then $x \models \varphi$ if and only if $x \in U$; for $\varphi = \neg\psi$, $x \models \varphi$ if and only if it is not the case that $x \models \psi$; for $\varphi = \bigwedge\{\psi_i : i \in \Lambda\}$, $x \models \varphi$ if and only if for every $i \in \Lambda$ we have $x \models \psi_i$.

For $F \subset \mathcal{L}_{\infty 0}(\mathcal{B})$ a countable set closed let $\tau(F)$ be the topology generated by $\mathcal{B}$ and all sets of the form $\{x \in X : x \models \varphi\}$, as φ ranges over F.

4.3 Lemma: *For $F \subset \mathcal{L}_{\omega_1 0}(\mathcal{B})$ a countable fragment, $\tau(F)$ forms the basis of a Polish topology on X.*

Proof: This follows by 4.1 and induction on the complexity of the infinitary sentences in F. If $\psi = \neg\phi$ it follows by inductive assumption and 4.1. For

$$\psi = \bigvee_{i \in \Lambda} \psi_i$$

it is trivial, since we are simply adding a new open set to the basis. At limit stages of the construction we may use that increasing countable unions of Polish topologies are again Polish—a classical fact that is recalled in [26] and [20]. ∎

Note then that if F is a fragment of $\mathcal{L}_{\infty 0}(\mathcal{B})$, then in any generic extension in which F becomes countable it must generate a Polish topology. We will frequently have cause to consider Polish spaces and continuous Polish groups both in the universe V and through future generic extensions. This is reasonable, since all the relevant statements of the form 'X is a Polish G-space' are $\underset{\sim}{\Pi}^1_1$, and hence absolute.

The next lemma merely makes the point that we may find the Vaught transform of a $\varphi \in \mathcal{L}_{\infty 0}(\mathcal{B})$ in a manner that is effective. The proof is by transfinite induction and can be viewed as a rephrasing of the usual proof that the Vaught transform of a Borel set is again Borel.

4.4 Lemma: *Let G be a Polish group, X a Polish G-space, $\mathcal{B}$ a countable basis for X, $\mathcal{B}_0$ a countable basis for G. Then to each $\varphi \in \mathcal{L}_{\infty 0}(\mathcal{B})$ and $V \in \mathcal{B}_0$ we may assign a formula $\varphi^{\Delta V} \in \mathcal{L}_{\infty 0}(\mathcal{B})$ such that:*

(i) $(V, \varphi) \mapsto \varphi^{\Delta V}$ *is uniformly* $\underset{\sim}{\Delta}_1$ *in any parameter coding* X, G, *the action, and the bases;*

(ii) *the fragment generated by* $\varphi^{\Delta V}$ *has the same cardinality as the fragment generated by* φ, *and in fact they have approximately the same logical complexity;*

(iii) *in all generic extensions* $V[H]$ *of* V *in which* $\varphi \in (\mathcal{L}_{\omega_1 0}(\mathcal{B}))^{V[H]}$ *we have*

$$\{x \in X \colon x \models \varphi^{\Delta V}\} = \{x \in X \colon \exists^* g \in V (g \cdot x \models \varphi)\}.$$

Note here that the calculation of whether $x \models \varphi$ is absolute to any model containing x and φ. The statement of 4.4 gives that the assignment $(V, \varphi) \mapsto \varphi^{\Delta V}$ will be $\underset{\sim}{\Delta}^1_2$ when restricted to $\varphi \in \mathcal{L}_{\omega_1 0}$, since $\underset{\sim}{\Delta}^{HC}_1 = \underset{\sim}{\Delta}^1_2$.

The next two lemmas again admit routine proofs by transfinite induction.

4.5 LEMMA: *Let* G, X, $\mathcal{B}$, $\mathcal{B}_0$ *be as in 4.4. Then to each* $\varphi \in \mathcal{L}_{\infty 0}(\mathcal{B})$ *and* $g \in G$ *we may assign a formula* $\varphi^g \in \mathcal{L}_{\infty 0}(\mathcal{B})$ *such that:*

(i) $(g, \varphi) \mapsto \varphi^g$ *is uniformly* $\underset{\sim}{\Delta}_1$ *in any parameter coding* X, G, *the action, and the bases;*

(ii) *the fragment generated by* φ^g *has the same cardinality as the fragment generated by* φ;

(iii) *in all generic extensions* $V[H]$ *of* V *in which* $\varphi \in (\mathcal{L}_{\omega_1 0}(\mathcal{B}))^{V[H]}$ *we have*

$$\{x \in X \colon x \models \varphi^g\} = \{x \in X \colon g \cdot x \models \varphi\}.$$

The next lemma can be contrasted with the notion of ∞-Borel from §2. In effect the lemma states that every ∞-Borel code is representable by some infinitary $\varphi \in \mathcal{L}_{\kappa 0}(\mathcal{B})$, for some ordinal κ.

4.6 LEMMA: *Let* X *be a Polish space and* $\mathcal{B}$ *be a basis. Then for* α *an ordinal,* $S \subset \alpha$, *and* $\psi \in \mathcal{L}(\in)$ *a formula in set theory, there is a corresponding* $\varphi(\alpha, S, \psi) \in \mathcal{L}_{\infty 0}(\mathcal{B})$ *such that*

(i) *in all generic extensions,*

$$\{x \in X \colon x \models \varphi(\alpha, S, \psi)\} = \{x \in X \colon L_\alpha[S, x] \models \psi(x, S)\};$$

(ii) *the transitive closure of* $\varphi(\alpha, S, \psi)$ *has cardinality* $|\alpha| + \aleph_0$, *so that for* $G \subset \mathrm{Coll}(\omega, \alpha)$ V*-generic,* $\varphi(\alpha, S, \psi) \in \mathcal{L}_{\omega_1 0}(\mathcal{B})$;

(iii) *the assignment* $\alpha, S, \psi \mapsto \varphi(\alpha, S, \psi)$ *is* Δ_1 *in any parameter coding the space and the basis.*

4.7 THEOREM (Becker–Kechris): *Let G be a Polish group and X a Polish G-space, $\mathcal{B}$ a basis for X, $\mathcal{B}_0$ a basis for G, $G_0 \subset G$ be a countable dense subgroup. Let $\mathcal{C}$ be a countable collection of Borel sets in X such that*

(i) *$\mathcal{C}$ is an algebra—in other words, closed under finite Boolean operations;*
(ii) *$\mathcal{C}$ is closed under translation by elements in G_0;*
(iii) *$\mathcal{C}$ is closed under Vaught transforms from $\mathcal{B}_0$, so that for $C \in \mathcal{C}$ and $V \in \mathcal{B}_0$, $C^{*V}, C^{\Delta V} \in \mathcal{C}$;*
(iv) *$\mathcal{C}$ forms the basis of a Polish topology on X.*

Then: $\{C^{\Delta V} \colon C \in \mathcal{C}, V \in \mathcal{B}_0\}$ forms the basis of a Polish topology on X under which it remains a Polish G-space. (See [3].)

The assumptions above guarantee that for any $C_1, \ldots, C_l \in \mathcal{C}$, $V_1, \ldots, V_l \in \mathcal{B}_0$, $C_1^{\Delta V_1} \cap \cdots \cap C_l^{\Delta V_l}$ is the union of sets in $\{C^{\Delta V} : C \in \mathcal{C}, V \in \mathcal{B}_0\}$: If

$$x \in C_1^{\Delta V_1} \cap \cdots \cap C_n^{\Delta V_n}$$

then we may find a basic open neighbourhood W of the identity and $g_1, \ldots, g_n$ in G_0 such that

$$x \in C_1^{*W g_1} \cap \cdots \cap C_n^{*W g_n}$$

and each

$$W g_i \subset V_i.$$

$$\therefore x \in (g_1^{-1} C_1)^{*W} \cap \cdots \cap (g_n^{-1} C_n)^{*W} \subset (g_1^{-1} C_1 \cap \cdots \cap g_n^{-1} C_n)^{\Delta W} \subset C_1^{\Delta V_1} \cap \cdots \cap C_n^{\Delta V_n}.$$

4.8 LEMMA: *Let X, $\mathcal{B}$ be as in 4.7, let E be a $\underset{\sim}{\Sigma}^1_2$ equivalence relation on X, and let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$, and σ a term for $V^{\mathbb{P}}$ a term for an element in X, such that*

$$(p, p) \Vdash_{\mathbb{P} \times \mathbb{P}} \sigma[\dot{G}_l] E \sigma[\dot{G}_r],$$

where, as usual, $\dot{G}_l$ and $\dot{G}_r$ refer to the generic objects on the left and right copies of $\mathbb{P}$.

Then there is a set $F \subset \mathcal{L}_{\infty 0}(\mathcal{B})$ and a $\varphi \in F$ such that:

(i) *$\tau(F)$ generates a Polish topology on X in any generic extension in which F becomes countable;*
(ii) *for any generic $H \subset \mathbb{P}$, $\sigma[H] \models \varphi$;*
(iii) *for any x_1, x_2 appearing in any generic extension of V with $x_1, x_2 \models \varphi$ we have $x_1 E x_2$.*

Proof:

CLAIM: *There exists* φ *so that*

$$x \models \varphi$$

if and only if there exists (in some generic extension of V*) a* V*-generic* $H \subset \mathbb{P}$ *with*

$$p \in H \quad \text{and} \quad \sigma[H] = x.$$

Proof of Claim: This is by considering the factor forcing. Following §25 of [18] we may find a forcing notion $\mathbb{Q}_0$ and a $\mathbb{Q}_0$-term σ_0 so that for any $H \subset \mathbb{P}$ that is V-generic there exists V-generic $H_0 \subset \mathbb{Q}_0$ with

$$\sigma_0[H_0] = \sigma[H] \quad \text{and} \quad V[\sigma_0[H_0]] = V[H_0]$$

and some $\mathbb{Q}_1 \in V[H_0]$ and $H_1 \subset \mathbb{Q}_1$ a $V[H_0]$-generic filter with

$$V[H_0][H_1] = V[H].$$

Now choose α to be a sufficiently large beth fixed point, $S \subset \alpha$ a code for $(V_\alpha, \mathbb{P}, \sigma)$ (so that $L_\alpha[S] = V_\alpha$), and let $\psi(\cdot,\cdot)$ be a formula so that

$$L_\alpha[x, S] \models \psi(x, S)$$

if and only if it satisfies there is some forcing notion $\mathbb{Q}_1$ whose generic object will make possible the construction of an $L_\alpha[S]$-generic $H \subset \mathbb{P}$ with

$$\sigma[H] = x \quad \text{and} \quad p \in H.$$

Thus we may apply 4.6. (Claim ∎)

By closing φ under subformulas and finite Boolean operations, we obtain a Polish topology by 4.3. Thus we have (i) and (ii).

So now suppose that $x_1, x_2 \models \varphi$. Then we can generically find H_1, H_2 that are V-generic below p with $\sigma[H_1] = x_1$, $\sigma[H_2] = x_2$. Then by choosing $H_3 \subset \mathbb{P}$ sufficiently generic below p, and setting $x_3 = \sigma[H_3]$, we get that $H_1 \times H_3$ and $H_1 \times H_2$ are both V-generic.

Then by the assumptions on $\mathbb{P}$, p, and σ,

$$x_1 E x_3, \quad x_2 E x_3,$$

and thus

$$x_1 E x_2. \qquad \blacksquare$$

In the context of Polish group actions 4.7 suggest a refinement.

4.9 COROLLARY: *Let G, X, $\mathcal{B}$, G_0 be as in 4.7, and let $\mathbb{P}$ $p \in \mathbb{P}$, and σ be as in 4.8. Then there is a set $F_0 \subset \mathcal{L}_{\infty 0}(\mathcal{B})$ and a $\varphi_0 \in F_0$ such that:*

(i) *$\tau(F_0)$ generates a Polish topology on X in any generic extension $V[H]$ in which F_0 becomes countable,* and *$(X, \tau(F_0))$ remains a Polish G-space;*

(ii) *for any generic $H \subset \mathbb{P}$*

$$\forall g \in G(g \cdot \sigma[H] \models \varphi_0);$$

(iii) *for any x_1, x_2 in any generic extension of V with $x_1, x_2 \models \varphi_0$ we have $x_1 E_G^X x_2$.*

Proof: First let $\mathbb{P}^*$ be the forcing notion of $\mathbb{P}$ followed by the version of Cohen forcing obtained by using the basic open sets in G to create a generic group element. Then let τ be the term in $\mathbb{P}^*$ for $\dot{g} \cdot \sigma[\dot{H}]$, where $\dot{g}$ names the generic group element and $\dot{H}$ denotes the generic on $\mathbb{P}$, and let $q = \langle p, 1\rangle$ be the condition in $\mathbb{P}^*$ obtained by insisting that p be in the generic $\dot{H}$.

Then $\mathbb{P}^*$, $q \in \mathbb{P}^*$, and τ continue to satisfy the assumptions of 4.8, but we have engineered the further result that if (H, h) is a generic on $\mathbb{P}^*$, and $x = \tau[(H, h)]$, then in any future generic extension in which $(2^{\mathbb{P}^*})^V$ becomes countable we have that $\forall^* g \in G((H, gh)$ is V generic for $\mathbb{P}^*$ below $\langle p, 1\rangle)$.

Now we choose F and φ as in 4.8, for $\mathbb{P}^*$ and τ, but taking enough care to ensure closure under G_0 translation and Δ-Vaught transforms with respect to $\mathcal{B}_0$. This can certainly be achieved by 4.4 and 4.5. So then we obtain (i), (ii) and (iii) as in 4.8, but with the further condition that in any generic extension of V containing x in which $(2^{\mathbb{P}^*})^V$ becomes countable

$$x \models \varphi \Rightarrow \exists^* g \in G(g \cdot x \models \varphi),$$

and so in the notation of 4.4,

$$x \models \varphi \Rightarrow x \models \varphi^{\Delta G}.$$

CLAIM: *In all generic extensions, $\varphi^{\Delta G}$ is G-invariant.*

Note that for any generic extension in which φ is in $\mathcal{L}_{\omega_1 0}(\mathcal{B})$ we will have that $x \models \varphi^{\Delta G}$ if and only if there is a non-meager collection of group elements $g \in G$ such that $g \cdot x \models \varphi$. Thus

$$x \models \varphi^{\Delta G}$$

if and only if

$$g_0 \cdot x \models \varphi^{\Delta G}$$

for any $g_0 \in G$, in this model, and hence also in $V[x, g_0]$—since the calculation of

$$g_0 \cdot x \models \varphi^{\Delta G}$$

is absolute to $V[x, g_0]$. (Claim ∎)

So if we now follow 4.7 and let F_0 be $\{\psi^{\Delta U}: \psi \in F, U \in \mathcal{B}_0\}$, then we can take $\varphi_0 = \varphi^{\Delta G} \in F_0$. Then this is as required since $\varphi^{\Delta G}$ is G-invariant. ∎

5. Becker's theorem

5.1 Lemma: *Let G be a Polish group admitting a left complete invariant metric, and X a Polish G-space, and $\mathcal{B}$ a countable basis for X. Let $x \in X$. Let $\mathcal{M}$ be a class inner model of* ZF + DC, *with X, G, and the action existing in $\mathcal{M}$, in the sense of being coded by a parameter in $\mathcal{M}$. Let $F \subset \mathcal{L}_{\infty 0}(\mathcal{B})$ be a countable set such that*

(i) *$F \in \mathcal{M}$;*
(ii) *in some generic extension of V, $\tau(F)$ generates a Polish topology on X, including the original topology, with $(X, \tau(F))$ a Polish G-space;*
(iii) *$[x]_G$ is $\tau(F)$-open.*

Then: $[x]_G \bigcap \mathcal{M} \neq \emptyset$.

Proof: [The idea of the proof is that much of the crucial information regarding the orbit $[x]_G$ can already be determined in $\mathcal{M}$; if $H \subset \mathrm{Coll}(\omega, F)$ is $\mathcal{M}$-generic, then $\mathcal{M}[H] \bigcap [x]_G \neq \emptyset$ and, for instance, $\{\varphi \in F: \exists x_0 \in [x]_G (x_0 \models \varphi)\} \in \mathcal{M}$; the task is to use this information to find a representative of the orbit. Normally this would be quite out of the question, since for arbitrary Polish groups we may have orbits that are 'generic' in the sense of 4.8 over an inner model without there having a representative. Here the complete left invariant metric enables us to obtain a representative solely from the relatively weak information regarding how the map $g \mapsto g \cdot x$ is $\tau(F)$-open. As preshadowed by some arguments in [1], this is a little like back-and-forth constructions in model theory but only here the restrictive assumption on the group implies that the 'forth' direction alone is sufficient to entail isomorphism.]

Note that the natural map from

$$G \twoheadrightarrow [x]_G,$$

$$g \mapsto g \cdot x,$$

is $\tau(F)$-open and $\tau(F)$-continuous by 1.9 and (ii) above in the assumptions of the lemma.

Let $\mathcal{B}_0$ be a countable basis for G in $\mathcal{M}$. Let d_r be a right invariant complete metric on G. Let d be a complete metric on X. Then for any $V \in \mathcal{B}_0$ with $1_G \in V$ we define

$$\mathcal{B}(V)=\{\varphi \in F\colon \forall x_0, x_1 \in [x]_G \forall \varphi' \in F((x_0 \models \varphi)\wedge(x_1 \models \varphi\wedge\varphi') \Rightarrow x_0 \models (\varphi')^{\Delta V})\}.$$

CLAIM (1): *For any neighbourhood $V \in \mathcal{B}_0$ of 1_G and $x_0 \in [x]_G$ there is some $\varphi \in \mathcal{B}(V)$ with $x_0 \models \varphi$.*

Proof of Claim: Since $[x]_G$ is open with respect to $\tau(F)$ it is certainly in $\underset{\sim}{\Pi}^0_2(X, \tau(F))$. Now the claim follows by 1.10. (Claim (1) ∎)

CLAIM (2): *The function*

$$V \mapsto \mathcal{B}(V)$$

is an element of $\mathcal{M}$.

Proof of Claim: Note that this function is an invariant of $[x]_G$. By assumption on x we may find $\varphi_0 \in F$ so that in all generic extensions $\mathcal{M}[H]$ of $\mathcal{M}$ in which F becomes countable

$$M[H] \cap [x]_G \neq \emptyset \quad \forall x_0 \in M[H](xE_G x_0 \Leftrightarrow x_0 \models \varphi_0).$$

Note then any such $\mathcal{M}[H]$ will have the function

$$V \mapsto \mathcal{B}(V)$$

by the absoluteness of $\underset{\sim}{\Pi}^1_1$.

Thus $\mathcal{B}(V)$ will be uniformly definable over $\mathcal{M}$ as the set of $\varphi \in F$ such that for all forcing notions $\mathbb{P} \in \mathcal{M}$

$$\mathcal{M}^{\mathbb{P}} \models \forall \varphi' \in F \forall x_0, x_1 \in X((x_0 \models \varphi_0 \wedge \varphi) \wedge (x_1 \models \varphi_0 \wedge \varphi \wedge \varphi') \Rightarrow x_0 \models (\varphi')^{\Delta V}).$$

(Claim (2) ∎)

Thus using DC in $\mathcal{M}$ we may find sequences

$$(V_n)_{n\in\mathbb{N}} \subset \mathcal{B}_0, \quad (U_n)_{n\in\mathbb{N}} \subset \mathcal{B}, \quad \varphi_n \in \mathcal{B}(V_n) \quad \text{and} \quad (z_n)_{n\in\mathbb{N}} \subset X$$

such that if $H \subset \mathrm{Coll}(\omega, F)$ is $\mathcal{M}$-generic then $\mathcal{M}[H]$ satisfies

$$d_r(V_n) < 2^{-n}, \quad d(U_n) < 2^{-n},$$
$$\exists x_n \in [x]_G \cap U_{n+1}(x_n \models \varphi_n), \quad z_n \in U_n,$$
$$y \in X(y \models \varphi_{n+1} \Rightarrow y \in U_{n+1} \wedge y \models \varphi_n), \quad \overline{U}_{n+1} \subset U_n.$$

The above assignments exist already in $\mathcal{M}$ since it has access to the function $V \mapsto \mathcal{B}(V)$. Since $(z_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{M}$, we can find $z_\infty \in \mathcal{M}$ such that $z_n \to z_\infty$ as $n \to \infty$.

Meanwhile in $\mathcal{M}[H]$ use the definition of $\mathcal{B}(V_n)$ to find $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in [x]_G \cap U_n$ and $x_n \models \varphi_n$ and such that at each n there is

$$g_n \cdot x_n = x_{n+1}.$$

Note that $d_r(g_n, 1_G) < 2^{-n}$ implies $d_r(g_n \cdot g_{n-1} \cdot \dots \cdot g_0, g_{n-1} \cdot \dots \cdot g_0) < 2^{-n}$. Thus if we set $h_n = g_n \cdot g_{n-1} \cdot \dots \cdot g_0$ then $(h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in G with respect to d_r. So there is some h_∞ that is the limit of $(h_n)_{n\in\mathbb{N}}$. Then let

$$x_\infty = h_\infty \cdot x_0 = \lim\nolimits_{\mathbb{N}} x_n.$$

Since $d(x_n, z_n) < 2^{-n}$ we get

$$x_\infty = z_\infty \in \mathcal{M} \cap [x]_G,$$

as required. ∎

5.2 Corollary: *Let $\mathcal{M}$ be a class inner model of* ZF + DC. *Let X be a Polish G-space, with both objects along with the action existing in $\mathcal{M}$. Let $(\mathbb{P}, \sigma, p) \in \mathcal{M}$ such that σ is term for the $\mathcal{M}^{\mathbb{P}}$ such that*

$$(p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l] E^X_G \sigma[\dot{G}_r].$$

Then there is some $y \in \mathcal{M}$ such that

$$p \Vdash_{\mathbb{P}} \sigma[\dot{G}] E^X_G y.$$

Proof: We may as well assume that our universe V has a representative of the generic equivalence class, since otherwise we may replace V by $V[H]$ for some suitably generic H. Then the theorem follows by 5.1 and 4.9. ∎

It is concluded from the results of [6] that 5.2 characterizes when a closed subgroup of S_∞ is cli, in that if G is a closed subgroup of the symmetric group that does not admit a left invariant complete metric then there is a Polish G-space X and σ a term for the forcing notion $\mathbb{P}$ =Coll(ω, ω_1) such that

$$\mathbb{P} \times \mathbb{P} \Vdash \sigma[\dot{G}_l] E^X_G \sigma[\dot{G}_r],$$

and for all $x \in X$

$$\mathbb{P} \Vdash \neg(\sigma[\dot{G}] E^X_G x).$$

5.3 Theorem: TVC(cli, $\underset{\sim}{\Sigma}^1_1$)—*which is to say, if G is a cli Polish group, X a Polish G-space, $A \subset X$ $\underset{\sim}{\Sigma}^1_1$, then either $|A/G| \leq \aleph_0$ or there is a perfect set $P \subset A$ such that any two elements in P are E^X_G-inequivalent.*

Proof: Suppose for a contradiction that A is $\underset{\sim}{\Sigma}^1_1$ and has uncountably many orbits but not perfectly many.

Claim (1): *In all generic extensions A has uncountably many orbits.*

Proof of Claim (as in [26]): Let $\mathcal{F}(G)$ denote the standard Borel space of all closed subsets of G with the Effros Borel structure. The statement that A has uncountably many orbits may be recast as

$$\forall (x_i)_{i\in\omega} \in X \forall (F_i)_{i\in\omega} \in \mathcal{F}(G)(\exists i \in \omega(F_i \neq G_{x_i} \vee \exists a \in A \setminus \bigcup_{i\in\omega}[x_i]_{i\in\omega})),$$

where G_{x_i} denotes the stabilizer of x_i. Since the set

$$\{(x, F) \in X \times \mathcal{F}(G) : F = G_x\}$$

is $\underset{\sim}{\Pi}^1_1$ and any $[x]_G$ is uniformly $\underset{\sim}{\Delta}^1_1$ in any code for G_x the whole display above is $\underset{\sim}{\Pi}^1_2$, and hence absolute by Shoenfield. (Claim (1) ∎)

Thus for $\mathbb{P}$ =Coll$(\omega, (2^{\aleph_0}))$, there will some term σ with

$$\mathbb{P} \Vdash \forall x \in V(x \notin [\sigma[\dot{G}]]_G).$$

Claim (2): *In no generic extension does A have a perfect set of inequivalent orbits.*

Proof of Claim (again as in [26]): Suppose in some generic extension we have perfect $P \subset A$ so that for all $x, y \in P$

$$xE_G y \Rightarrow x = y.$$

Suppose $A = \{x \in X : \exists y \in \omega^\omega((x, y) \in B)\}$ for some Borel $B \subset X \times \omega^\omega$. Following 18.A of [20] we may use Jankov von Neumann uniformization to find a

C-measurable function $f: P \to \omega^\omega$ so that for all $x \in P$ the point $f(x)$ witnesses $x \in A$. Since any C-measurable function must be Baire measurable and hence continuous on a relatively comeager $Q \subset P$ and any such comeager set must in turn have a perfect subset, we may find perfect $P_0 \subset P$ and continuous f_0 with domain P_0 such that for any $x \in P_0$, $f_0(x)$ witnesses $x \in A$. But then the statement that there exists perfect $P_0 \subset X$ and continuous $f_0: P_0 \to \omega^\omega$ such that

$$\forall x \in P_0((x, f_0(x)) \in B)$$

$$\forall x, y \in P_0(xE_G y \Rightarrow x = y)$$

will be $\underset{\sim}{\Sigma}^1_2$, and hence absolute. (Claim (2) ∎)

CLAIM (3): *There exists some condition $p \in \mathbb{P}$ that decides the equivalence class, in the sense that*

$$(p,p) \Vdash_{\mathbb{P}\times\mathbb{P}} \sigma[\dot{G}_l] E^X_G \sigma[\dot{G}_r].$$

Proof of Claim (as in [9], [10], or [28]): Or else in the generic extension $V^{\mathrm{Coll}(\omega, 2^{|\mathbb{P}|})}$ choose $(D_n)_{n\in\mathbb{N}}$ an enumeration of the open dense subsets of $\mathbb{P}\times\mathbb{P}$, and then choose $(p_s)_{s\in 2^{<\mathbb{N}}}$ so that for $s \neq t \in 2^n$,

$$(p_s, p_t) \in D_n,$$

and

$$(p_{s0}, p_{s1}) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg(\sigma[\dot{G}_l] E^X_G \sigma[\dot{G}_r]);$$

then for $w \in 2^{\mathbb{N}}$ we let G_w be the filter generated by $\{p_{w|n} : n \in \mathbb{N}\}$ and obtain for any two distinct $w, w' \in 2^{\mathbb{N}}$ that $\sigma[G_w]$ and $\sigma[G_{w'}]$ are inequivalent; by meeting all the dense sets in V we will obtain that the function

$$2^{\mathbb{N}} \to A$$

$$w \mapsto \sigma[G_w]$$

is continuous and hence we finish with a perfect set of orbit inequivalent points in A. This contradicts Claim (2). (Claim (3) ∎)

So suppose instead that there is a condition p deciding the equivalence class. For $H \subset \mathbb{P}$ V-generic, $x = \sigma[H]$, in $V[H]$ we can apply 5.2 to $\mathcal{M} = V$, and obtain that $[x]_G$ has a representative in V, contradicting the assumption on $\mathbb{P}$ and σ. ∎

Perhaps the main difference between the proof of 5.3 and the arguments of [1] is in the use of the 'virtual' Polish topologies described in §4. This has proved a useful technical tool, since recently it has been possible to algebraically describe those Polish groups for which the Vaught conjecture holds on analytic sets: TVC$(G, \underset{\sim}{\Sigma}^1_1)$ if and only if no closed subgroup of G has S_∞ as a continuous homomorphic image.

5.4 THEOREM: *Let G be a Polish group with a left invariant complete metric acting continuously on a Polish space X, and let $A \subset X$ be $\underset{\sim}{\Sigma}^1_1$. Then either*

(I) *there is an absolutely $\underset{\sim}{\Delta}^1_2$ function $\theta: A \to 2^\omega$ such that for all $x_1, x_2 \in A$*

$$\exists g \in G(g \cdot x_1 = x_2) \Leftrightarrow \theta(x_1) = \theta(x_2),$$

or

(II) *there is a Borel $\theta: \mathbb{R} \to A$ such that for all $r_1, r_2 \in \mathbb{R}$*

$$r_1 - r_2 \in \mathbb{Q} \Leftrightarrow \exists g \in G(g \cdot \theta(r_1) = \theta(r_2)).$$

Proof: Let $A = \{x \in X : \exists y \in \omega^\omega B(x,y)\}$, for some Polish space $B \subset X \times \omega^\omega$. Define E on B by $(x_1,y_1)E(x_2,y_2)$ if and only if $x_1 E^X_G x_2$. This is a $\underset{\sim}{\Sigma}^1_1$ equivalence relation such that through all generic extensions every equivalence class is Borel.

Following 1.11, one case is that $E_0 \sqsubseteq_c E$, when we are quickly finished. Alternatively, we obtain a C-measurable in the codes reduction into $2^{<\omega_1}$, call it θ. Then it is $\underset{\sim}{\Pi}^1_2$ to assert that

$$\forall x_1, x_2 \in X(\theta(x_1) = \theta(x_2) \Rightarrow x_1 E^X_G x_2)$$

and thus absolute. Let z be a real coding the action, and any parameters used in the definition of θ.

Then for all $(x,y) \in B$ there must be a representative of the E-equivalence class of (x,y) in any generic extension of $L[\theta(x,y), z]$ in which $\theta(x,y)$ is countable, by absoluteness of $\underset{\sim}{\Sigma}^1_2$. Thus $[x]_G$ will be generic over $L[\theta(x,y), z]$, and thus by 5.2 there will be some $x_0 \in [x]_G \bigcap L[\theta(x,y), z]$.

So now we can define θ_0: $A \to X$ by letting $\theta_0(x)$ be the first real under the canonical wellorder in $L[\theta(x,y), z]$ with $x E^X_G x_0$. This gives a reduction of $E^X_G|_A$ to $\mathrm{id}(X)$, which can in turn be reorganized to give a reduction into $\mathrm{id}(2^\omega)$ or $\mathrm{id}(\mathbb{R})$.

It should be clear from the definitions that the function θ_0 is $\underset{\sim}{\Delta}^1_2$. Since every step in its definition appealed to facts that are absolute between the universe V and its generic extensions, the function is absolutely $\underset{\sim}{\Delta}^1_2$. ∎

Combining these ideas with the methods of §3 it can be shown that:

5.5 THEOREM ($AD^{L(\mathbb{R})}$): *Let G be a cli Polish group and let X be a Polish G-space. Let $A \subset X$ be in $L(\mathbb{R})$. Then either*

(I) $E_G^X|_A \leq \mathrm{id}(2^\omega)$,

or

(II) $E_0 \sqsubseteq_c E_G^X|_A$.

The observation that underlies the proof of 5.5 is that if we are in case (I) of 3.3, as witnessed by

$$\theta\colon A \to 2^{<\omega_1}$$

in $L(\mathbb{R})$, and if $S \subset \alpha$ is an ∞-Borel code for

$$\{(x, w_0, w_1, i)\colon x \in X,\ w_0, w_1 \text{ code } \alpha < \delta,\ \theta(x) \in 2^\delta,\ \theta(x)(\alpha) = i\},$$

then $\theta(x) \in L[S, x]$; in the notation of the proof of 3.2, $[x]_G$ will be generic over $M_S^x[\theta(x)]$; thus

$$[x]_G \bigcap M_S^x[\theta(x)] \neq \emptyset$$

by 5.1.

Finally, no cli group can code countable sets of reals. Since this is one of the simplest equivalence relations induced by the symmetric group, and in some ways appears distinctive of this group, the result underscores the divergence between cli group actions and arbitrary orbit equivalence relations induced by S_∞.

5.6 THEOREM: *Let $Y = \mathbb{R}^{\mathbb{N}}$ and define E by $(y_n)_{n\in\mathbb{N}} E (x_n)_{n\in\mathbb{N}}$ if and only if $\{y_n\colon n \in \mathbb{N}\} = \{x_n\colon n \in \mathbb{N}\}$—so this is the orbit equivalence relation induced by the S_∞-action of $(g \cdot \vec{x})(n) = x(g^{-1}(n))$ for $\vec{x} \in Y$ and $g \in S_\infty$. Then there is no cli Polish group G Polish G-space X with $E \leq_B E_G^X$.*

Proof: Instead suppose $\theta\colon Y \to X$ performs a Borel reduction. Note that this statement is $\underset{\sim}{\Pi}_2^1$:

$$\forall y_1, y_2 \in Y (y_1 E y_2 \Leftrightarrow \exists g \in G(g \cdot \theta(y_1) = \theta(y_2));$$

and hence it would be absolute through all generic extensions.

Let $\mathbb{P}$ be the forcing to collapse $2^{\aleph_0}$ to ω, and let $\sigma[\dot{G}]$ denote the term in $V^{\mathbb{P}}$, an element of Y, that enumerates every real once. Thus

$$\mathbb{P} \times \mathbb{P} \Vdash \sigma[\dot{G}_l] E \sigma[\dot{G}_r].$$

Thus if we let $\sigma_0[\dot{G}]$ be the term for $\theta(\sigma[\dot{G}])$, by the absoluteness of the assumptions on θ

$$(\mathbb{P} \times \mathbb{P} \Vdash \sigma_0[\dot{G}_l] E_G^X \sigma_0[\dot{G}_r]).$$

Thus by 5.2 there is some $x \in X$ with

$$\mathbb{P} \Vdash \sigma_0[\dot{G}] E_G^X x.$$

However

$$V^{\mathbb{P}} \models \exists y \in Y(\theta(y) E_G^X x),$$

and so this must hold already in V by the absoluteness of $\underset{\sim}{\Sigma}_2^1$. So fix $y \in Y$ with $\theta(y) E_G^X x$. Then again by the absoluteness of the assumptions on θ, $\mathbb{P} \Vdash \sigma[\dot{G}] E y$.

This is absurd, since any such y would need to enumerate $\mathbb{R}$ in order type ω. ∎

While a similar result is proved for abelian groups in [13] without an appeal to metamathematics, the only known proof of 5.6 uses forcing.

6. Knight's model

6.1 Definition: Let $\sigma \in \mathcal{L}_{\omega_1\omega}$, for $\mathcal{L}$ some countable language, which we may assume without loss of generality to be relational. Then Mod(σ) is the set of all models of σ whose underlying set is $\mathbb{N}$. We let $\tau(\sigma)$ be the topology generated by sets of the form $\{M \in \text{Mod}(\sigma): M \models \psi(n_0, n_1, ..., n_k)\}$ where $(n_0, \ldots, n_k)$ is a finite sequence of natural numbers and ψ is a formula in the fragment generated by σ, in the sense that it is in the smallest collection of formulas containing σ and closed under subformulas, substitutions, and the first order operations of negation, finite disjunction, finite conjunction, and existential quantifiers. Mod($\mathcal{L}$) is the collection of $\mathcal{L}$ models on $\mathbb{N}$ with the topology generated by first order logic.

We then let S_∞ act on Mod(σ) by

$$(g \cdot M) \models R(n_0, \ldots, n_k) \Leftrightarrow M \models R(g^{-1}(n_0), \ldots, g^{-1}(n_k)),$$

for any $R \in \mathcal{L}$, $(n_0, \ldots, n_k)$ a finite sequence in $\mathbb{N}$. The equivalence relation E_{S_∞} induced by this action on Mod(σ) is frequently denoted by $\cong |\text{Mod}(\sigma)$.

6.2 Lemma (see [7]): *For any $\sigma \in \mathcal{L}_{\omega_1\omega}$, $(\text{Mod}(\sigma), \tau(\sigma))$ is a Polish S_∞-space.*

6.3 Definition: For M a model and $\vec{a} \in M^{<\omega}$ one defines the canonical α type of $\vec{a}$, $\varphi_\alpha^{\vec{a},M} \in \mathcal{L}_{\infty\omega}$, by induction on α: $\varphi_0^{\vec{a},M}$ is the infinitary formula expressing the quantifier free type of $\vec{a}$ in M;

$$\varphi_{\alpha+1}^{\vec{a},M} = \varphi_\alpha^{\vec{a},M} \bigwedge_{b \in M} \exists x \varphi_\alpha^{\vec{a}b,M} \wedge \forall x \bigvee_{b \in M} \varphi_\alpha^{\vec{a}b,M}.$$

At limit stages we take intersections.

The **Scott height** of M is the least γ such that for all $\vec{a}$, $\varphi_\delta^{\vec{a},M}$ determines $\varphi_{\delta+1}^{\vec{a},M}$. The **Scott sentence** of M, $\varphi_M \in \mathcal{L}_{\infty\omega}$ states what γ-types exist for γ the Scott height and that this *is* the Scott height. As in [22], two countable models are isomorphic if and only if they have the same Scott sentence.

6.4 Definition: Let M be a countable model with underlying set $\mathbb{N}$. Then $\mathrm{Aut}(M) = \{g \in S_\infty : g \cdot M = M\}$.

6.5 THEOREM (folklore): *Let G be a subgroup of S_∞. Then G is closed in S_∞ if and only if $G = \mathrm{Aut}(M)$ for some countable M with underlying set $\mathbb{N}$.*

The authors of [3] noticed that this allows a curious analogue in the context of Polish group actions.

6.6 THEOREM (Becker–Kechris): *Let $G = \mathrm{Aut}(M)$ be a closed subgroup of S_∞; let $\mathcal{L}$ be the language of M. Let X be a Polish G-space. Then there is a language $\mathcal{L}' \supset \mathcal{L}$ extending the language of M and $\sigma \in \mathcal{L}'_{\omega_1\omega}$ such that $\sigma \models \varphi_M$ and $|X/G|$ is Borel equivalent to $\mathrm{Mod}(\sigma)$, in the sense that there are θ: $X \to \mathrm{Mod}(\sigma)$ and ρ: $\mathrm{Mod}(\sigma) \to X$ such that:*

(i) *θ witnesses $E_G^X \leq_B \cong |\mathrm{Mod}(\sigma)$;*
(ii) *ρ witnesses $\cong |\mathrm{Mod}(\sigma) \leq_B E_G^X$; and*
(iii) *these are orbit inverses to one another in the sense that for all $x \in X$, $xE_G^X(\rho \circ \theta(x))$.*

Proof (sketch): Let $(\mathcal{O}_m)_{m\in\mathbb{N}}$ be a countable basis for X. We may associate to each $x \in X$ the model M_x, with relations $(R_{m,k})_{m,k\in\mathbb{N}}$, where for $(n_1, \ldots, n_k)$ a k-tuple in $\mathbb{N}$,

$$M_x \models R_{m,k}(n_1, \ldots, n_k) \Leftrightarrow \forall^* g \in G(g(n_1) = 0 \wedge \cdots \wedge g(n_k) = k-1 \Rightarrow g \cdot x \in \mathcal{O}_m).$$

It is shown in the course of [3] that for $x_1, x_2 \in X$, $x_1 E_G^X x_2$ if and only if there is some $g \in G$ with $g \cdot M_{x_1} = M_{x_2}$.

At this point we may define N_x to be the expansion of M_x obtained by incorporating all the relations of M. Since any $g \in G$ fixes M_x, we then obtain that $N_{x_1} \cong N_{x_2}$ if and only if $\exists g \in G(g \cdot M_{x_1} = M_{x_2})$. We let $\mathcal{L}'$ be the language of these model N_x. Since $\{g \cdot N_x : g \in S_\infty\}$ is a Borel S_∞ set, we may characterize it as the model of some $\sigma \in \mathcal{L}'_{\omega_1\omega}$. ∎

6.7 THEOREM (Gao): *Let $G = \mathrm{Aut}(M)$ be a closed subgroup of S_∞, with $\mathcal{L}$ the language of M. Then G is cli if and only if every $\mathcal{L}_{\omega_1\omega}$ elementary embedding $\pi\colon M \to M$ is onto.*

6.8 THEOREM (Knight): *There is a countable model M with language $\{<, f_0, f_1, \ldots\}$, where*

(i) *$<$ is a linear ordering on M;*
(ii) *each f_n is unary function;*
(iii) *for each $y \in M$, $\{x \in M\colon x < y\} = \{f_n(y)\colon n \in \omega\}$; and*
(iv) *there is a non-onto $\mathcal{L}_{\omega_1\omega}$ elementary embedding from M to M.*

6.9 LEMMA: *Let G be a Polish group, X a Polish G-space, $A \subset X$ a counterexample to $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$. Then for each ordinal δ there exists a sequence $(\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha)_{\alpha\in\delta}$ so that for each $\alpha < \beta < \delta$*

$$(p_\alpha, p_\alpha) \Vdash_{\mathbb{P}_\alpha\times\mathbb{P}_\alpha} \sigma_\alpha[\dot{G}_l] E_G \sigma_\alpha[\dot{G}_r];$$
$$(p_\alpha, p_\beta) \Vdash_{\mathbb{P}_\alpha\times\mathbb{P}_\beta} \neg(\sigma_\alpha[\dot{G}_l] E_G \sigma_\beta[\dot{G}_r]).$$

Proof: As in the proof of 5.3, for $\mathbb{P}_0 = \mathrm{Coll}(\omega, \kappa)$, κ sufficiently big, there will some term σ_0 for $\mathbb{P}_0$ and $p_0 \in \mathbb{P}_0$ with

$$\mathbb{P}_0 \Vdash \forall x \in V(x \notin [\sigma_0[\dot{G}]]_G),$$
$$(p_0, p_0) \Vdash_{\mathbb{P}_0\times\mathbb{P}_0} \sigma_0[\dot{G}_l] E^X_G \sigma_0[\dot{G}_r].$$

By applying this argument again in $V^{\mathbb{P}_0}$ we may find $\mathbb{P}_1$ and σ_1 such that

$$\mathbb{P}_1 \Vdash \forall x \in V(x \notin [\sigma_0[\dot{G}]]_G),$$
$$(p_1, p_1) \Vdash_{\mathbb{P}_1\times\mathbb{P}_1} \sigma_1[\dot{G}_l] E^X_G \sigma_1[\dot{G}_r],$$
$$\mathbb{P}_1 \times \mathbb{P}_0 \Vdash \sigma_1[\dot{G}_l] \notin [\sigma_0[\dot{G}_r]]_G.$$

Continuing this transfinitely we may find $(\sigma_\alpha, \mathbb{P}_\alpha, p_\alpha)$ for $\alpha < \delta$ as described in the conclusion of the lemma. ∎

6.10 THEOREM: *There is a Polish group G such that:*

(i) $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$*—in the sense that if $A \subset X$ is $\underset{\sim}{\Sigma}^1_1$, and X is a Polish G-space, then either $|A/G| \le \aleph_0$ or there is a perfect set $P \subset A$ of inequivalent reals;*

and

(ii) *G is not cli.*

Proof: Let $G = \mathrm{Aut}(M)$ for M as in 6.8. G is not cli by 6.8(iv) and 6.7. Suppose for a contradiction that $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$ fails, so let X be a Polish G-space, $A \subset X$

with exactly $\aleph_1$ many orbits, and fix $\theta\colon X \to \mathrm{Mod}(\varphi)$ for some countable $\varphi \in \mathcal{L}_{\omega_1\omega}$ implying the Scott sentence of M, $\mathcal{L} \supset \{<, f_0, f_1, \ldots\}$.

Thus for $\delta = |H(\omega_2)|^+$ we may find a sequence $(\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha)_{\alpha\in\delta}$ as in 6.9. In particular, for any such α and $H_\alpha \subset \mathbb{P}_\alpha$ V-generic below p_α, the equivalence class of $\sigma_\alpha[H_\alpha]$ does not depend on the choice of H_α. Thus the isomorphism type of $\theta(\sigma_\alpha[H_\alpha])$ is independent of the choice of the generic, and thus so too the Scott sentence. Hence, as in §1 of [14], an induction on the set theoretical rank shows that the Scott sentence φ_α of $\theta(\sigma_\alpha[H_\alpha])$ exists in V.

Let $\gamma(\alpha)$ be the Scott height of any model of φ_α (where this model, as opposed to its Scott sentence, may only exist in a generic extension). Let A_α be the collection of canonical $\gamma(\alpha)$ types realized by any such model; note that the cardinality of $\gamma(\alpha)$ is less than or equal to that of $A_\alpha \times A_\alpha$, since for any $\beta < \gamma(\alpha)$ there must exist distinct canonical $\gamma(\alpha)$ types whose βth approximations are equal but whose $\beta + 1$th approximations diverge. Thus, with the possible exception of the case that it is finite, the cardinality of A_α must be at least that of the transitive closure of φ_α. Hence by assumption of δ we may find φ_α whose transitive closure has cardinality strictly greater than $\aleph_1$ and hence for which A_α has cardinality greater than or equal to $\aleph_2$.

Fix this α. We can define a quasi-linear ordering on A_α by $\varphi' \leq \varphi''$ if and only if for any model N of $\varphi(\alpha)$ and $\vec{a}, \vec{b} \in N$ with $\varphi^{\vec{a},N}_{\gamma(\alpha)} = \varphi'$ and $\varphi^{\vec{b},N}_{\gamma(\alpha)} = \varphi''$, for all $a_0 \in \vec{a}$ there is some $b_0 \in \vec{b}$ and some $c \in N$ with

$$N \models c < b_0, \quad \varphi^{a_0,N}_{\gamma(\alpha)} = \varphi^{c,N}_{\gamma(\alpha)}.$$

Since each model of φ is an expansion of the Knight model, it follows by 6.8(iii) that for each $\varphi' \in A(\alpha)$ there are at most countably many $\varphi'' \leq \varphi'$, and we have a contradiction. ∎

A positive answer to the next question would help clarify 6.10.

6.11 Question: If $G = \mathrm{Aut}(M)$, for M a countable model, and $\mathrm{TVC}(G, \underset{\sim}{\Sigma}^1_1)$ fails, then must the Scott sentence of M have a model of size $2^{\aleph_0}$?

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